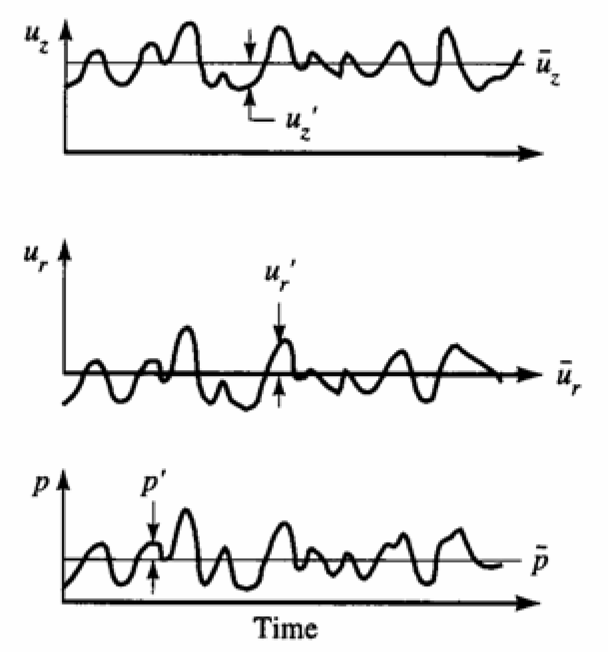
Chapter 9

Turbulent flow and boundary layer

laminar

turbulent

$$u_z\big|_{r=0} = \frac{2Q}{\pi R^2}$$



$$u_r\big|_{r=0} = 0$$

$$p\big|_{z=L/2} = \frac{4\mu LQ}{\pi R^4}$$

Random fluctuating funtions of time

Characterized by mean, intensity, frequency spectrum

$$\bar{u}_z \equiv \frac{1}{\tau}\int_t^{t+\tau} u_z\, dt$$

Time averaging

$$\overline{S} \equiv \frac{1}{\tau}\int_t^{t+\tau} S\,dt \qquad \overline{S} \equiv \frac{1}{\tau}\int_t^{t+\tau} (\overline{S} + S')\,dt = \overline{\overline{S}} + \overline{S'} \qquad \overline{S'} \equiv 0$$

$$\overline{TS} \equiv \frac{1}{\tau}\int_t^{t+\tau} (\overline{T} + T')(\overline{S} + S')\,dt = \overline{\overline{T}\,\overline{S}} + \overline{T'\overline{S}} + \overline{\overline{T}S'} + \overline{T'S'}$$

$$= \overline{T}\,\overline{S} + \overline{T'S'}$$

$$u_z = \overline{u}_z + u'_z \qquad \frac{\partial \overline{u}_x}{\partial x} + \frac{\partial \overline{u}_y}{\partial y} + \frac{\partial \overline{u}_z}{\partial z} = 0$$

$$u_x = \overline{u}_x + u'_x$$

$$u_y = \overline{u}_y + u'_y \qquad \frac{\partial}{\partial t}\rho\overline{u}_z + \frac{\partial}{\partial x}\rho\overline{u}_z\overline{u}_x + \frac{\partial}{\partial y}\rho\overline{u}_z\overline{u}_y + \frac{\partial}{\partial z}\rho\overline{u}_z\overline{u}_z = -\frac{\partial\overline{p}}{\partial z} + \mu\nabla^2\overline{u}_z + \rho g_z$$

$$p = \overline{p} + p'$$

$$-\frac{\partial}{\partial x}\rho\overline{u'_z u'_x} - \frac{\partial}{\partial y}\rho\overline{u'_z u'_y} - \frac{\partial}{\partial z}\rho\overline{u'_z u'_z}$$

Reynolds stress

Convection of momentum by velocity fluctuations

$$\overline{T}_{zz} = \rho\overline{u'_z u'_z}, \quad \overline{T}_{zx} = \rho\overline{u'_z u'_x}, \ldots$$

Reynolds stress models

Newtonian

$$T_{12} = -\mu\Delta_{12}$$

turbulent

$$\overline{T}_{zx} = -\mu_t \frac{\partial \overline{u}_z}{\partial x}$$

Turbulent viscosity parameter; not a fluid property

Prandtl's mixing length theory $\overline{T}_{zx} = -\rho l^2 \left|\frac{\partial \overline{u}_z}{\partial x}\right| \left(\frac{\partial \overline{u}_z}{\partial x}\right)$

Turbulent flow in a pipe

Assume; 1.mean flow is fully developed and steady state

2.mixing length is a linear function of distance from the wall

$$\overline{u}_z = \overline{u}_z(r)$$

$$\overline{u}_r = \overline{u}_\theta = 0$$

$$l = \kappa s \qquad \text{where} \qquad s \equiv R - r$$

$$\Rightarrow \quad \overline{T}_{rz} = \rho\kappa^2 s^2 \left(\frac{\partial \overline{u}_z}{\partial s}\right)^2$$

Axial component of time-averaged Navier-Stokes equation

$$0 = \frac{\Delta \overline{P}}{L} + \mu \frac{1}{r}\frac{d}{dr}\left(r \frac{d\overline{u}_z}{dr} \right) - \frac{1}{r}\frac{d}{dr}(r\overline{T}_{rz})$$

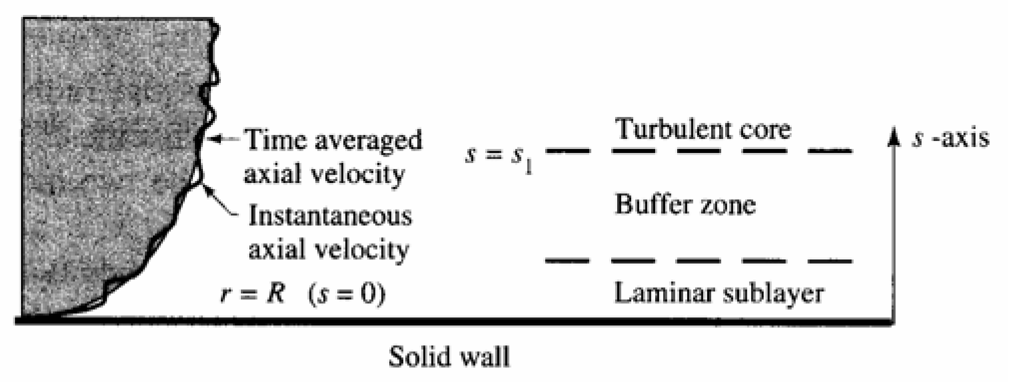
1. Far from the wall; turbulence dominates viscous shear stress

$$\overline{T}_{rz} = \frac{\Delta \overline{P} R}{2L}\frac{r}{R} = \underline{T_{\mathrm{R}}}\frac{r}{R}$$

Time averaged shear stress at wall

$$\overline{T}_{rz} = \rho \kappa^2 s^2 \left(\frac{\partial \overline{u}_z}{\partial s} \right)^2 \qquad \rho \kappa^2 s^2 \left(\frac{\partial \overline{u}_z}{\partial s} \right)^2 = T_{\mathrm{R}}\left(1 - \frac{s}{R} \right)$$

$$\frac{\partial \overline{u}_z}{\partial s} = \left(\frac{T_{\mathrm{R}}}{\rho \kappa^2} \right)^{1/2} \frac{(1 - \underline{s/R})^{1/2}}{s}$$



Integrating from $s=s_1$ to $s>s_1$

$$u^+ = u_1^+ + \frac{1}{\kappa}\ln\frac{s^+}{s_1^+} \qquad u^+ \equiv \frac{\bar{u}_z}{u_*} \quad \text{and} \quad s^+ \equiv \frac{s u_* \rho}{\mu} \qquad u_* \equiv \left(\frac{T_R}{\rho}\right)^{1/2}$$

$$u^+ = u_1^+ - \frac{1}{\kappa}\ln s_1^+ + \frac{1}{\kappa}\ln s^+$$

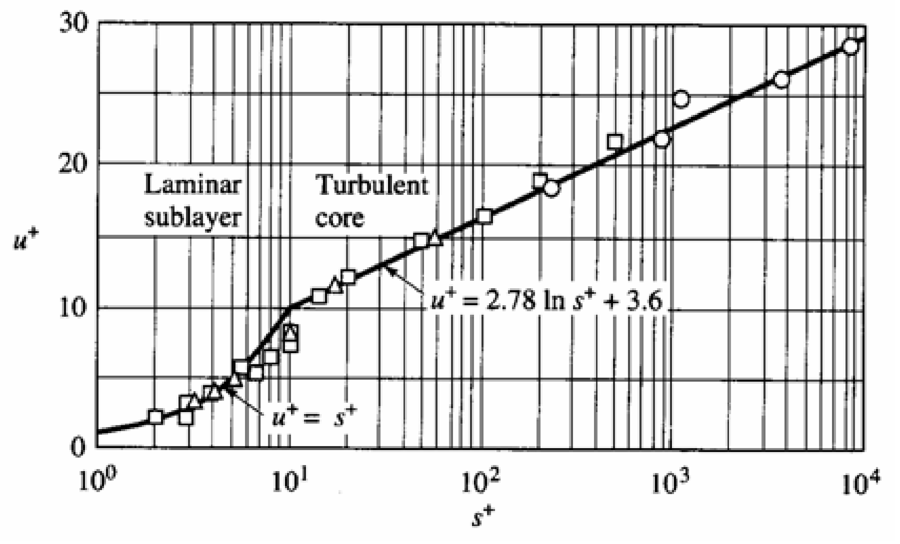
$$= A + B\ln s^+ \qquad u^+ = 2.78\ln s^+ + 3.6$$

2. Near the wall; viscous effect dominates

$$0 = \frac{\Delta\bar{P}}{L} + \mu\frac{1}{r}\frac{d}{dr}\left(r\frac{d\bar{u}_z}{dr}\right) - \frac{1}{r}\frac{d}{dr}(r\bar{T}_{rz})$$

$$\bar{T}_R = \frac{\Delta\bar{P}R}{2L} = \mu\frac{d\bar{u}_z}{ds}\bigg|_{s=0}$$

$$1 = \frac{du^+}{ds^+}\bigg|_{s=0} \qquad u^+ = s^+$$



Turbulent friction factor

$$f \equiv \frac{T_R}{\frac{1}{2}\rho U^2} = \frac{\Delta\bar{P} R}{\rho U^2 L} \qquad u_*^2 \equiv \frac{T_R}{\rho} = \frac{fU^2}{2} \qquad \frac{u_*^2}{U^2} = \frac{f}{2}$$

$$\pi R^2 U = \int_0^R 2\pi r \bar{u}_z \, dr$$

$$u^+ = 2.78 \ln s^+ + 3.6$$

$$\frac{1}{\sqrt{f}} = 1.97 \ln(\mathrm{Re}\sqrt{f}) - 2.45 = 4.52 \log(\mathrm{Re}\sqrt{f}) - 2.45$$

Neglected buffer zone, laminar sublayer

$$\frac{1}{\sqrt{f}} = 4 \log(\mathrm{Re}\sqrt{f}) - 0.4$$

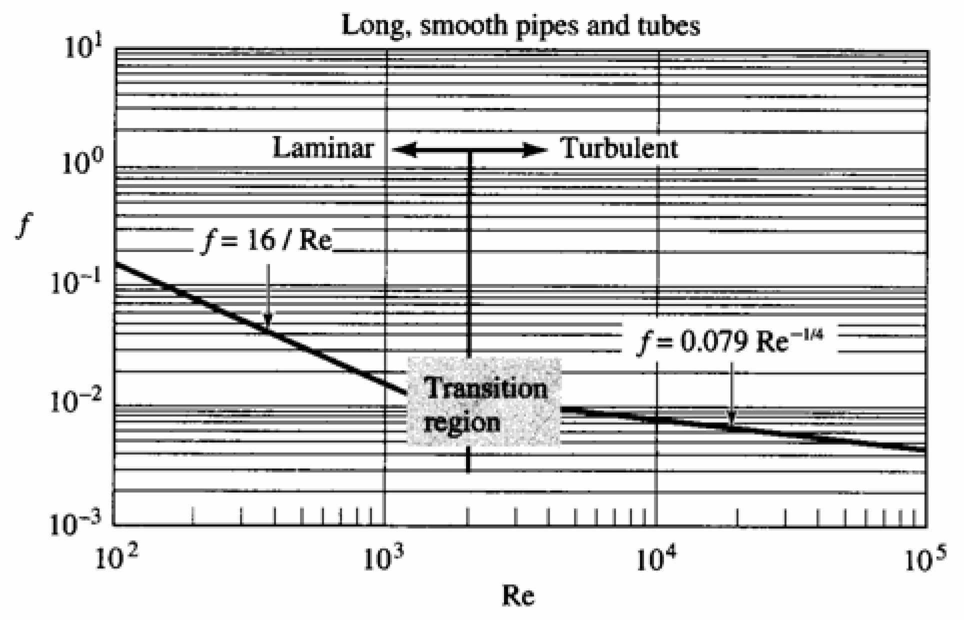
Semi-empirical, but works well

Friction factor for pipe flow

Laminar $\frac{\Delta P}{L} = \frac{8\mu Q}{\pi R^4} = \frac{8\mu U}{R^2}$ $f_{\text{lam}} = \frac{\Delta P R}{\rho U^2 L} = \frac{8\mu}{\rho U R} = \frac{16}{\text{Re}}$

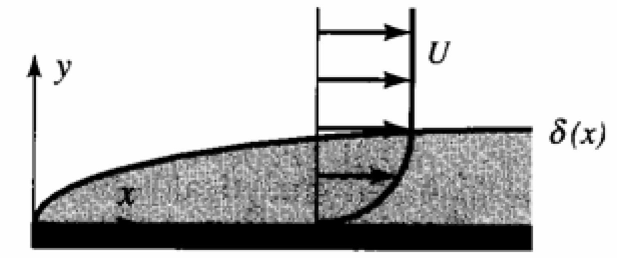
Hagen-Poiseuille

Turbulent $\frac{1}{\sqrt{f}} = 4\log(\text{Re}\sqrt{f}) - 0.4$ $f_{\text{lam}} = 0.079\,\text{Re}^{-0.25}$



Laminar boundary layer

No matter how turbulent the flow is far from the surface



$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0$$

$$\rho\left(u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}\right)$$

$$u_y=0;\quad \frac{\partial p}{\partial y}=0 \quad \text{within the boundary layer}$$

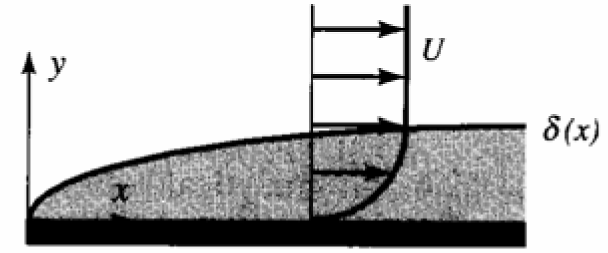
$$\rho\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)=-\cancel{\frac{\partial p}{\partial x}}+\mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}\right)$$

Negligible viscous effect outside the boundary layer

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}=0 \qquad \rho\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)=\mu\frac{\partial^2 u_x}{\partial y^2}$$

$$\boxed{\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y}-\frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2}=\nu\frac{\partial^3 \psi}{\partial y^3}}$$

Boundary conditions



$$u_x=u_y=0 \quad \text{on} \quad y=0, \text{ for } x>0$$

$$u_x=U \quad \text{for} \quad x<0$$

$$u_x\to U \quad \text{for} \quad y\to\infty$$

$$\frac{\partial \psi}{\partial y}=-\frac{\partial \psi}{\partial x}=0 \quad \text{on} \quad y=0, \text{ for } x>0$$

$$\frac{\partial \psi}{\partial y}\to U \quad \text{for} \quad x<0, \text{ and } y\to\infty$$

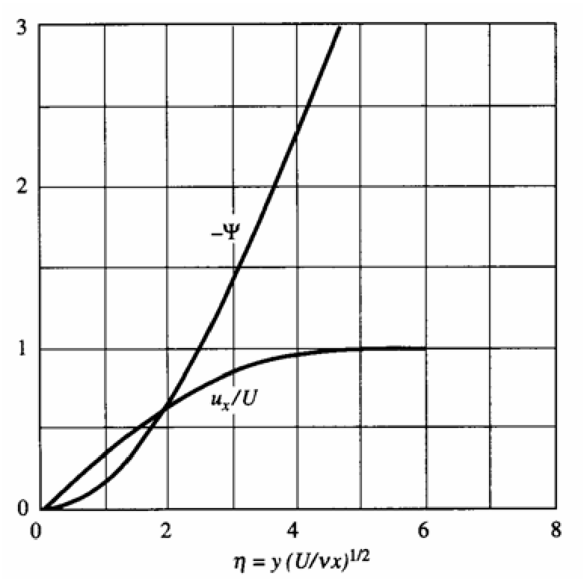
Blasius' similarity transformation

$$\Psi\equiv\frac{\psi}{(U\nu x)^{1/2}} \qquad \eta\equiv y\left(\frac{U}{\nu x}\right)^{1/2}$$

$$\boxed{2\Psi'''+\Psi\Psi''=0}$$

$$\Psi=\Psi'=0 \quad \text{at} \quad \eta=0$$

$$\Psi'\to 1 \quad \text{as} \quad \eta\to\infty$$



U_x becomes nearly U along the boundary layer $y = \delta(x)$

$$\delta \left(\frac{U}{\nu x} \right)^{1/2} = 5$$

Boundary layer thickness

Drag coeff.

$$C_D \equiv \frac{-\mu(\partial u_x / \partial y)_{y=0}}{\frac{1}{2}\rho U^2} \equiv 0.664 \left(\frac{\nu}{xU} \right)^{1/2} = \frac{0.664}{(\mathrm{Re}_x)^{1/2}}$$

Shear force

$$F_s = W \int_0^L -\mu \left(\frac{\partial u_x}{\partial y} \right)_{y=0} dx \qquad \frac{F_s / LW}{\frac{1}{2}\rho U^2} \equiv \overline{C}_D = \frac{1.328}{(UL/\nu)^{1/2}} = \frac{1.328}{(\mathrm{Re}_L)^{1/2}}$$

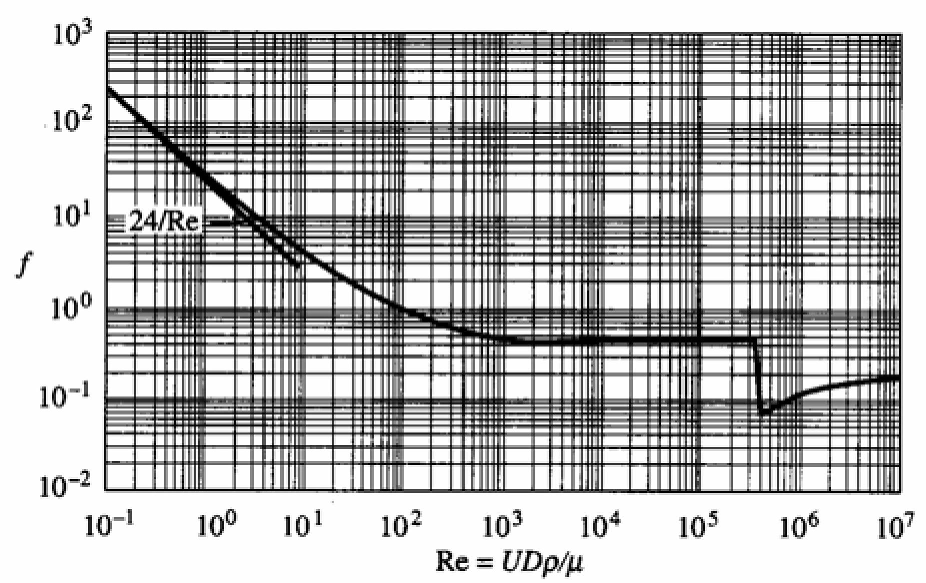
Local Reynolds number

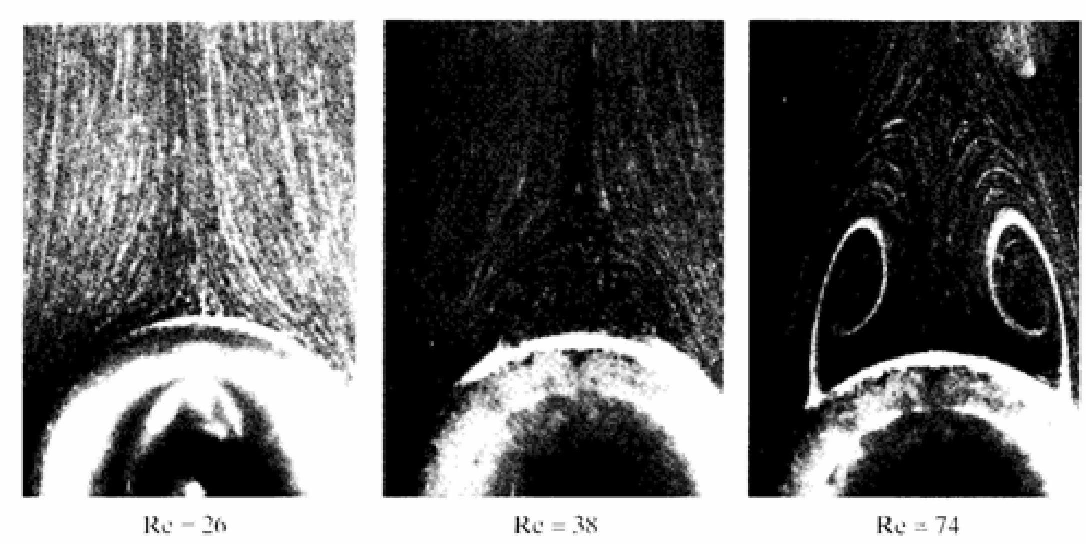
$$\mathrm{Re}_x \equiv \frac{xU}{\nu}$$

$$\mathrm{Re}_x^{\text{trans}} = 5 \times 10^5 \quad \text{(transition to turbulence)}$$

Turbulent drag force

$$f = C_D = \frac{\text{total force}}{\frac{1}{2}\rho U^2 \pi R^2}$$





Associated with transitions in the character of the boundary layer along the spherical surface