Chapter 7

Unsteady flows

Transient pressure flow

Assume: laminar flow at low Reynolds number, little effect of entrance region, isothermal incompressible Newtonian flow, the only velocity component $u_z(r,t)$

$$\rho\left(\frac{\partial u_z}{\partial t}+\cancel{u_r\frac{\partial u_z}{\partial r}}+\cancel{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}+\cancel{u_z\frac{\partial u_z}{\partial z}}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\cancel{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}}+\cancel{\frac{\partial^2 u_z}{\partial z^2}}\right)+\rho g_z \qquad \textbf{(4.3.24f)}$$

$$0=\frac{\partial u_z}{\partial z} \qquad \rho\frac{\partial u_z}{\partial t}=-\frac{\partial p}{\partial z}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)$$

Steady solution of Hagen-Poiseuille flow

$$u_z^s=u_z^{\max}\left[1-\left(\frac{r}{R}\right)^2\right] \qquad -\frac{\partial p}{\partial z}=\frac{\Delta p}{L}=\text{constant}=\frac{4u_z^{\max}\mu}{R^2}$$

Assume:

1.solution is the sum of steady solution and an unknown transient function *U(r,t)*

2.transient pressure is identical to the steady profile even for the unsteady flow

$$u_z=u_z^s+U(r,t) \qquad -\frac{\partial p}{\partial z}=\frac{\Delta p}{L}=\frac{\Delta P-\rho g_z L}{L}=\text{constant}$$

$$\rho\frac{\partial U}{\partial t} = \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) \qquad s = \frac{r}{R} \quad \Phi = \frac{U}{u_z^{\max}} \quad \tau = \frac{\mu t}{\rho R^2}$$

$$\frac{\partial \Phi}{\partial \tau} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial \Phi}{\partial s}\right)$$

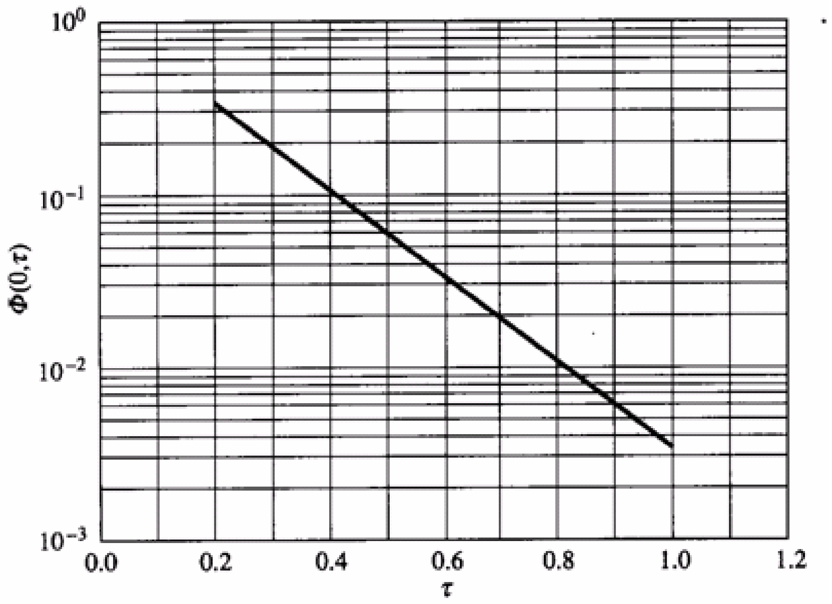
$$\frac{\partial \Phi}{\partial s} = 0 \quad \text{along} \quad s = 0 \qquad \Phi = -(1 - s^2) \quad \text{at} \quad \tau = 0 \qquad \Phi = 0 \quad \text{on} \quad s = 1$$

$$\boxed{\Phi = -8\sum_{n=1}^{\infty}\frac{J_o(\lambda_n s)}{\lambda_n^3 J_1(\lambda_n)}\exp(-\lambda_n^2 \tau)}$$

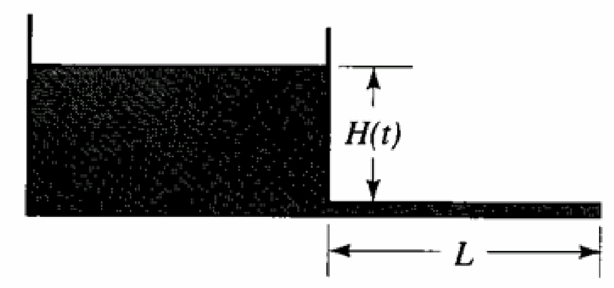
Transient disappears in a dimensionless time of the order of $\tau = 1$

$$t_\infty = \frac{\rho R^2}{\mu}$$

R: equivalent radius



Quasi-steady flows
- draining of a tank through a capillary



Time dependence comes from the time dependent pressure that drives the flow

$$p = \rho g H(t) \quad \text{at} \quad z = 0$$

Pressure source is hydrostatic head

$$\rho\left(\frac{\partial u_z}{\partial t} + \cancel{u_r \frac{\partial u_z}{\partial r}} + \cancel{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}} + \cancel{u_z \frac{\partial u_z}{\partial z}}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 u_z}{\partial z^2}}\right) + \rho g_z \quad \textbf{(4.3.24f)}$$

$$-\frac{\partial p}{\partial z} = C(t) = \frac{\rho g H(t)}{L} \qquad \rho\frac{\partial u_z}{\partial t} = \frac{\rho g H(t)}{L} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)$$

$$Q = -\frac{dV}{dt} = -A_T\frac{dH}{dt} = \int_0^R 2\pi r u_z \, dr$$

Coupled integro-differential equation

If the flow is slow enough, we use a steady state model for one particular feature of an unsteady but slowly varing flow

$$\boxed{u_z = -\frac{C(t)R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2\right]}$$

$$Q = \frac{\pi R^4}{8\mu}\frac{\rho g H(t)}{L} = -A_{\mathrm{T}}\frac{dH}{dt}$$

$$Q = \frac{\pi R^4}{8\mu}\frac{\Delta p}{L}$$

$$\frac{H}{H_{\mathrm{o}}} = e^{-\tau} \qquad \tau \equiv \frac{\pi\rho g R^4}{8\mu L A_{\mathrm{T}}}t$$

Requires infinite time for complete drainage

for 90% drainage

$$\tau = 2.3 \quad \text{when} \quad H/H_{\mathrm{o}} = 0.10$$

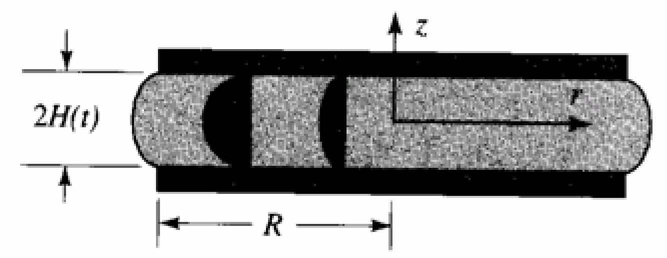
for 95% drainage

$$t_\infty \equiv \frac{24\mu L A_{\mathrm{T}}}{\pi\rho g R^4}$$

for 99% drainage

$$\tau = 4.6 \quad \text{when} \quad H/H_{\mathrm{o}} = 0.01$$

Squeezing flow



Time dependence comes from a time dependent change in the geometry

Laminar creeping flow

Order of magnitude

Axial $Q = 2\pi R^2 \dot{H}$

Radial $Q = 2\pi R \times 2H \times U_\mathrm{R}$

$$\frac{U_\mathrm{R}}{\dot{H}} = \frac{R}{2H}$$

$$\frac{R}{H} >> 1$$

Radial velocity is much greater than axial velocity (except very close to the plate)

$$\rho \frac{\partial u_z}{\partial t} = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_z}{\partial r} \right) + \frac{\partial^2 u_z}{\partial z^2} \right] \Rightarrow \frac{\partial p}{\partial z} = 0 \qquad p = p(r, t)$$

Since the disks are rigid bodies, u_z along the surfaces is independent of radial position -> assume u_z is everywhere independent of r

$$\frac{1}{r} \frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r}(r u_r) = -\frac{d u_z}{dz} = f(z, t) \qquad u_r = \tfrac{1}{2} r f(z, t)$$

$$\cancel{\rho \frac{\partial u_r}{\partial t}} = -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r}(r u_r) \right) + \frac{\partial^2 u_r}{\partial z^2} \right]$$

Assume viscous effect dominates inertial (accelerative) effect

$$\frac{dp}{dr} = \mu \frac{\partial^2 u_r}{\partial z^2} = \frac{\mu r}{2} \frac{\partial^2 f}{\partial z^2}$$

$$BC: \quad f = 0 \quad on \quad z = \pm H$$

$$u_r = \frac{rf}{2} = \frac{H^2}{2\mu} \left(-\frac{dp}{dr} \right) \left[1 - \left(\frac{z}{H} \right)^2 \right]$$

$$\int_{-H}^{H} 2\pi r u_r \, dz = 2\pi R^2 \dot{H} \qquad -\frac{1}{\mu r} \frac{dp}{dr} = \frac{3\dot{H}}{2H^3} \qquad p = \frac{3\dot{H}\mu R^2}{4H^3} \left[1 - \left(\frac{r}{R} \right)^2 \right]$$

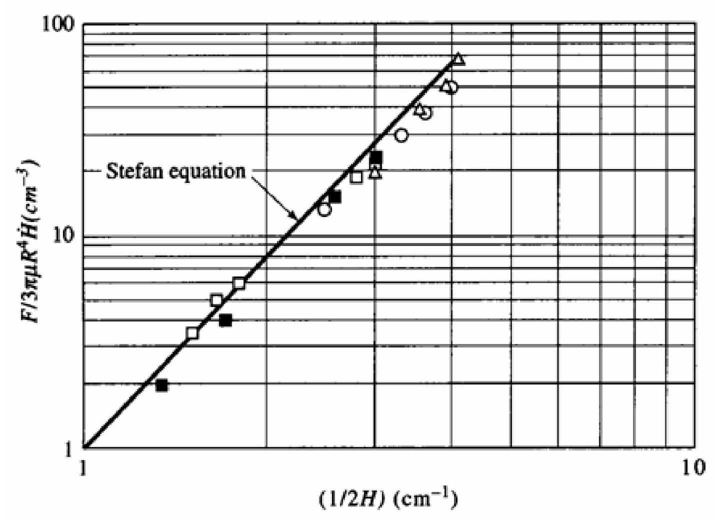
This pressure resists the movement of the disks toward each other, so an external force is required to drive them together

$$F = 2\pi \int_0^R T_{zz}\big|_{z=H} \, r\, dr$$

$$T_{zz} = p - 2\mu \frac{\partial u_z}{\partial z} \qquad 2\mu \frac{\partial u_z}{\partial z} = -2\mu \frac{1}{r}\frac{\partial}{\partial r}(r u_r) = \frac{3\mu \dot{H}}{H}\left[1 - \left(\frac{z}{H}\right)^2\right]$$

$$\boxed{F = 2\pi \int_0^R p(r) r\, dr = \frac{3\pi R^4 \mu \dot{H}}{8H^3}}$$

Stefan equation



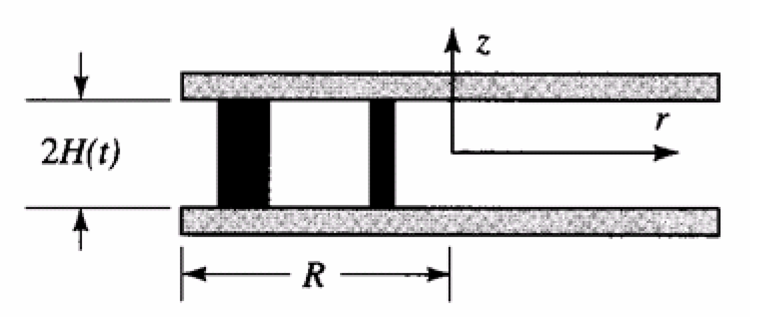
If the disks are driven under constant force

$$\dot{H} = -\frac{dH}{dt} = \frac{8F_o H^3}{3\pi R^4 \mu}$$

$$\frac{1}{H^2} - \frac{1}{H_o^2} = \frac{16 F_o t}{3\pi R^4 \mu}$$

good for high viscosity, slow squeezing

Squeezing flow of inviscid fluid



Slip boundary conditoin

Radial velocity is independent of the axial position

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{du_z}{dz} = C(t) \quad \Rightarrow \quad u_z = -\frac{\dot{H}}{H}z \qquad u_r = -\frac{\dot{H}}{2H}r$$

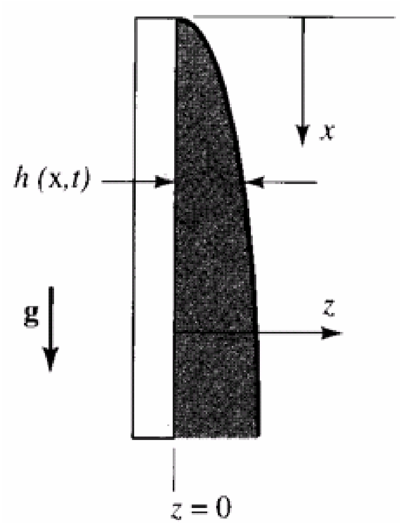
$$\rho\left[\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right] = -\frac{\partial p}{\partial r} \qquad -\frac{\partial p}{\partial r} = \frac{\rho r}{H}\left(\frac{\ddot{H}}{2} - \frac{\dot{H}^2}{4H}\right)$$

For constant speed squeezing

$$p = \frac{\rho r^2\dot{H}^2}{8H^2} \qquad F_1 = \int_0^R 2\pi r p(r)\, dr = \frac{\rho\pi R^4\dot{H}^2}{16H^2}$$

Strictly from the unsteady nature of the flow

Draining of a liquid film from a vertical plate



A uniform film of initial thickness H suddenly begins to drain, and the initial amount of liquid ultimately drains completely off the plate

Assume:
film thickness varies gradually in the x-direction (nearly parallel flow)
Viscous thin film with low Reynolds number (neglect inertia, surface tension)

$$\frac{\partial u}{\partial t} = \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial z^2}\right) + g$$

Lubrication approximation
almost parallel flow

$$u = \frac{g}{\nu}\left(hz - \frac{z^2}{2}\right)$$

Quasi-steady approximation:
the flow field for any film thickness *h(x,t)* is the same as the steady flow for uniform film thickness

The flow rate per unit width in the y-direction $$q = \int_0^{h(x,t)} u(z,t)\, dz = \frac{gh^3}{3\nu}$$

Any difference between the flow in and out of film must appear as a change in film thickness

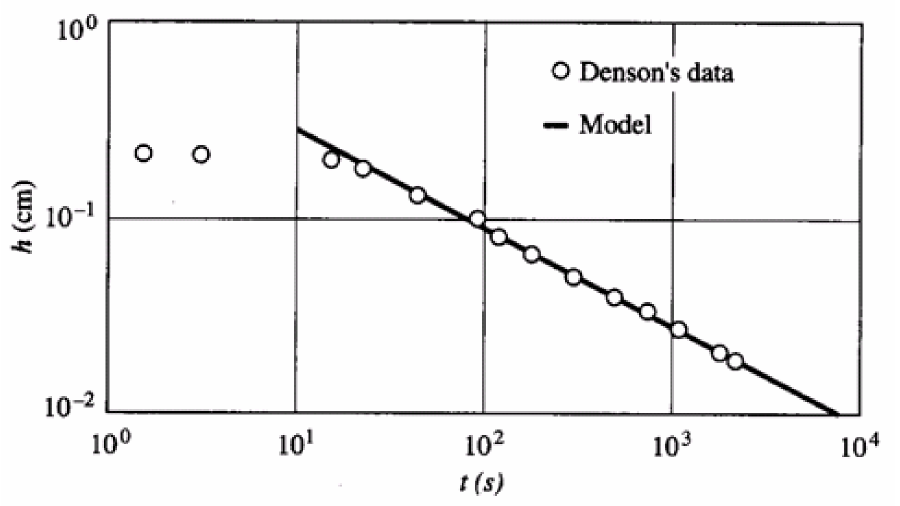
$$q_x - q_{x+dx} = \frac{\partial h}{\partial t} dx \qquad \frac{\partial h}{\partial t} = -\frac{\partial q}{\partial x} = -\frac{\partial}{\partial x}\frac{gh^3}{3\nu} = -\frac{gh^2}{\nu}\frac{\partial h}{\partial x}$$

$$h = Ax^\alpha t^\beta$$

Nonlinear partial differential equation

$$\beta A x^\alpha t^{\beta-1} = -\frac{g}{\nu}(A^2 x^{2\alpha} t^{2\beta})\alpha A x^{\alpha-1} t^\beta$$

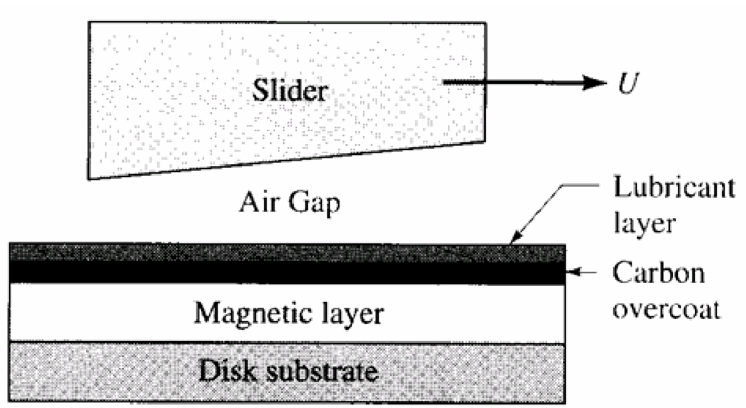
$$A = \left(\frac{\nu}{g}\right)^{1/2} \qquad \boxed{h = \left(\frac{\nu x}{gt}\right)^{1/2}}$$



1.Thickness is infinite at t=0
2.Thickness is zero at the top

Excellent at least after an initial period of time
as long as we are not concerned with the film thickness near the top plate

Leveling of a surface disturbance



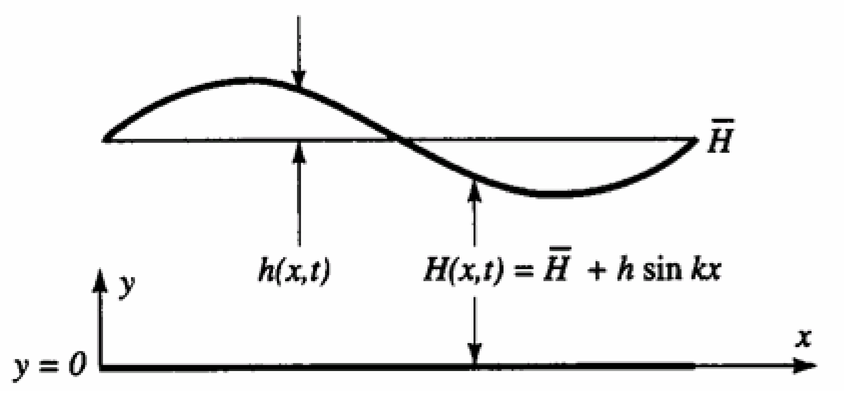
Magnetic recording system

Slider velocity ~ 10m/s
Air gap ~ a few hundred nanometers or less

Lubricant to prevent contact ~ 30-50Å

In the order of molecular size, certain phenomena are not accounted for by NS

Contact of the slider with the lubricant results in a furrow
-> how quickly the lubricant flows back into the furrow to restore the protection



Assume the disturbance to the film thickness is simusoidal

$$H(x) = \bar{H} + h \sin kx$$

$$k \equiv \frac{2\pi}{\lambda} \qquad h << \bar{H}$$

Assumption: lubrication approximation, quasi-steady state

$$0 = \mu\left(\cancel{\frac{\partial^2 u_x}{\partial x^2}} + \frac{\partial^2 u_x}{\partial y^2}\right) - \frac{\partial p}{\partial x} \qquad \frac{\partial p}{\partial y} = 0$$

Neglect any effect of gravity because the film is so thin

Surface tension provides the dominant force for restoration of the uniform film

$$p(x,t) = \frac{\sigma}{R(x,t)} \qquad \frac{1}{R} = -\frac{d^2 H}{dx^2} \qquad p = -\sigma H'' = \sigma h k^2 \sin kx \qquad p'' = -\sigma h k^4 \sin kx$$

$$u_x = \frac{p'}{2\mu} y^2 + \frac{a}{\mu} y + b \qquad p' = \partial p / \partial x$$

$$u_x = 0 \quad \text{at} \quad y = 0$$

$$\mu \frac{\partial u_x}{\partial y} = 0 \quad \text{at} \quad y = \underline{H(x,t) \approx \overline{H}}$$

$$\boxed{u_x = \frac{p'}{2\mu} y^2 - \frac{p' \overline{H}}{\mu} y}$$

$$Q = \int_0^{H(x,t)} u_x \, dy \approx \int_0^{\overline{H}} u_x \, dy = -\frac{p' \overline{H}^3}{3\mu} \qquad \frac{dQ}{dx} = -\frac{p'' \overline{H}^3}{3\mu} = -\frac{\partial H}{\partial t}$$

$$\frac{\partial H}{\partial t} = \frac{\partial h}{\partial t}\sin kx = -\frac{dQ}{dx} = \frac{p''\overline{H}^3}{3\mu} = -\frac{\sigma h k^4 \sin kx \overline{H}^3}{3\mu}$$

$$\frac{1}{h}\frac{\partial h}{\partial t} = -\frac{\sigma k^4 \overline{H}^3}{3\mu} \equiv -\beta$$

$$\boxed{h = h_o \exp(-\beta t)}$$

Decay rate for the disturbance

A disturbance to an extremely thin film will decay very slowly

If one uses a lubricant of low viscosity,

1.Low viscosity lubricants will have a smaller load-bearing capacity, and it will be easier for the head to crash through the lubricant

2.Since the high centrigugal force tends to produce a radial flow of lubricant off the disk

-> design of new lubricants is important (i.e. that bond chemically to the topmost solid layer of the disk