Chapter 5

Dimensonal analysis

Dynamic similarity

$$\nabla \cdot \mathbf{u} = 0 \qquad \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}$$

$$x^* = x/L, \quad \mathbf{u}^* = \mathbf{u}/U, \quad t^* = Ut/L, \quad p^* = p/\rho U^2 \qquad \nabla \equiv \frac{\partial}{\partial \mathbf{x}} = \frac{1}{L}\frac{\partial}{\partial \mathbf{x}^*}$$

$$\nabla^* \cdot \mathbf{u}^* = 0 \qquad \frac{\partial \mathbf{u}^*}{\partial t^*} + \mathbf{u}^*\cdot\nabla^*\mathbf{u}^* = -\nabla^* p^* + \frac{1}{\mathrm{Re}}\nabla^{*2}\mathbf{u}^* + \frac{1}{\mathrm{Fr}}\frac{\mathbf{g}}{g}$$

$$\boxed{\mathrm{Re} = \frac{\rho U L}{\mu}} \qquad \boxed{\mathrm{Fr} = \frac{U^2}{gL}}$$

Boundary conditions

$$u_x = 0 \quad \text{on} \quad y = 0 \qquad\qquad u_x^* = 0 \quad \text{on} \quad y^* = 0$$

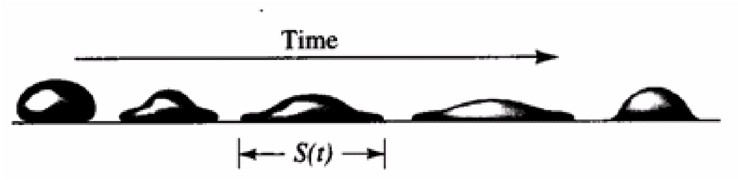
$$u_x = U \quad \text{on} \quad y = B \qquad\qquad u_x^* = 1 \quad \text{on} \quad y^* = \frac{B}{L}$$

$$p - \mu\Delta_{nn} - \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = 0 \qquad\qquad p^* - \frac{1}{\mathrm{Re}}\Delta^*_{nn} - \frac{1}{\mathrm{We}}\left(\frac{1}{R_1^*} + \frac{1}{R_2^*}\right) = 0$$

$$\boxed{\mathrm{We} = \frac{\rho U^2 L}{\sigma}}$$

If two flows occur in geometrically similar systems in which viscous effects and interfacial phenomena occur, and if *Re, We, Fr* are the same in both systems, the two systems are dynamically similar

Inkjet printing



Goal: to know how much a drop spreads on impact with a surface

$$V = 8\,nanoliters = 8\times10^{-12}m^3, \quad \mu = 0.005Pa\cdot s, \quad \sigma = 0.04N/m, \quad U = 1m/s$$

We want to design a scaled-up version of this process to facilitate observation and measurement of the drop dynamics (designing a dynamically similar system)

Length scale $L = V^{1/3} = (8\times10^{-12})^{1/3}\,\mathrm{m} = 2\times10^{-4}\,\mathrm{m}$

Velocity scale $U = 1\,\mathrm{m/s}$

Assume gravity is not a significant factor (this may not be the case)
-> ignore *Froude* number

$$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{1000(1)0.0002}{0.005} = 40 \qquad \mathrm{We} = \frac{\rho U^2 L}{\sigma} = \frac{1000(1)^2(0.0002)}{0.04} = 5$$

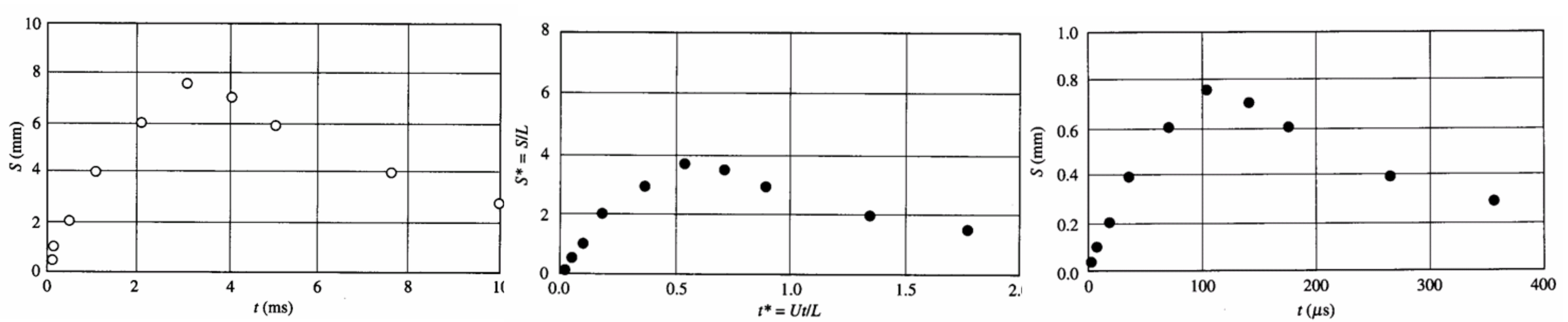
We must design an experiment such that

$$\mathrm{Re} = 40 = \left(\frac{\rho U L}{\mu}\right)_{\exp} \qquad \mathrm{We} = 5 = \left(\frac{\rho U^2 L}{\sigma}\right)_{\exp}$$

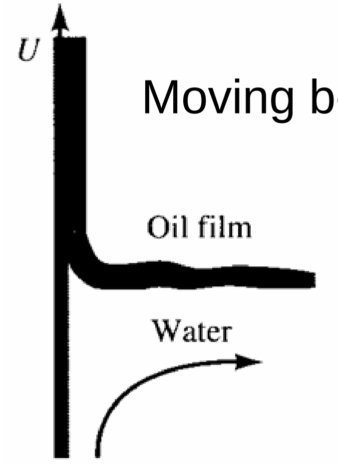
5 parameters, 2 constrains -> 3 free parameters

$$\sigma = 0.05\,\mathrm{N/m}, \quad \rho = 1000\,\mathrm{kg/m^3}, \quad \underline{L = 0.002m}$$

$$\Rightarrow \quad U = 0.354\,\mathrm{m/s}, \quad \mu = 0.018\ \mathrm{Pa \cdot s}$$



Removing oil from water surface



Moving belt with hydrophobic surface

Goal: design an experimental model to learn more about how the rate of oil entrainment is related to operating parameters such as belt speed and physical properties

Dynamics are dependent on the balance between inertial, viscous, surface, gravitational forces

$$\mathrm{Re} = \left(\frac{\rho U L}{\mu}\right)_{\text{model}} = \left(\frac{\rho U L}{\mu}\right)_{\text{system}} \quad \mathrm{We} = \left(\frac{\rho U^2 L}{\sigma}\right)_{\text{model}} = \left(\frac{\rho U^2 L}{\sigma}\right)_{\text{system}} \quad \mathrm{Fr} = \left(\frac{U^2}{gL}\right)_{\text{model}} = \left(\frac{U^2}{gL}\right)_{\text{system}}$$

5 parameters, 3 constraints -> 2 free parameters

$$L_{\text{model}} = kL_{\text{system}} \qquad \rho_{\text{model}} = \rho_{\text{system}}$$

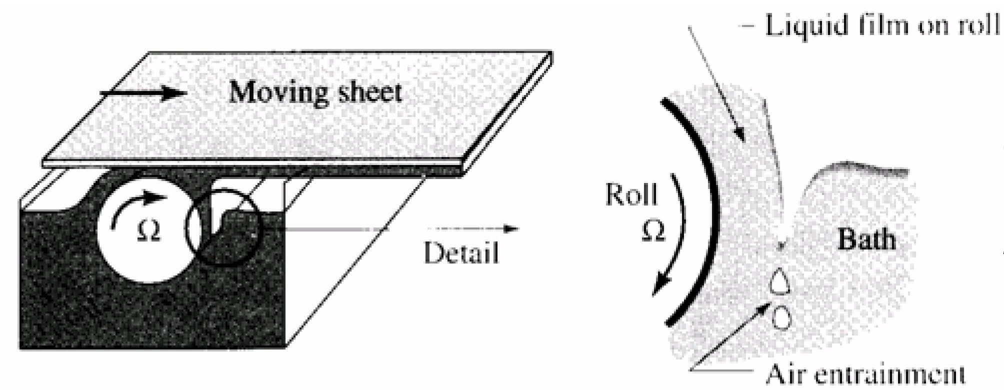
$$U_{\text{model}} = k^{1/2} U_{\text{system}}$$

$$\mu_{\text{model}} = k^{3/2} \mu_{\text{system}}$$

$$\sigma_{\text{model}} = k^{2} \sigma_{\text{system}}$$

1. Large k does not meet surface tension
2. Cannot take all three dimensionless group together
3. Need more physics to make knowledgeable evaluations of the relative importance of the three dynamic groups, *Re, We, Fr*

A roll coating system



Critical roll speed *U** for air entrainment

H/L ~ 0.01; roll appears as a plane

No characteristic length scale

$$\frac{\mathrm{We}}{\mathrm{Re}} = \frac{\mu U}{\sigma} = \mathrm{Ca}$$

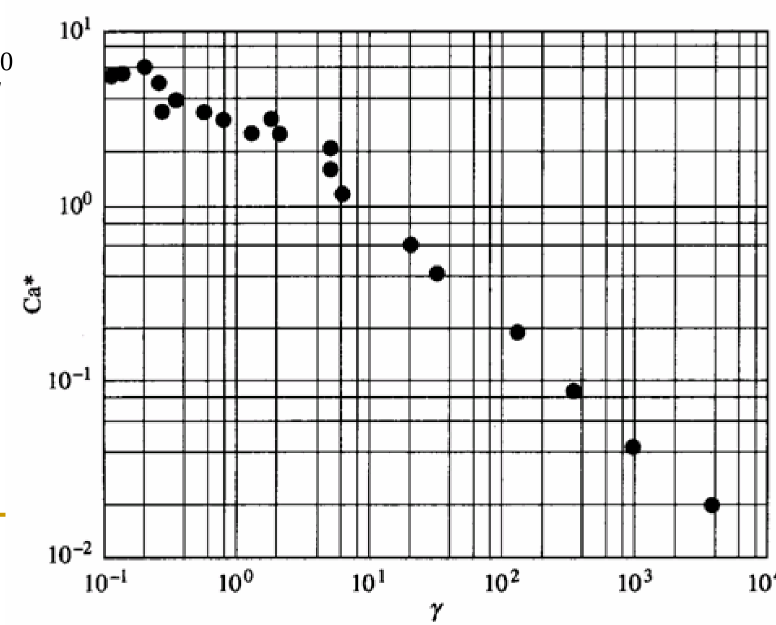
Ca*; dimensionless critical entrainment velocity

Predict Ca* from a knowledge of physical properties

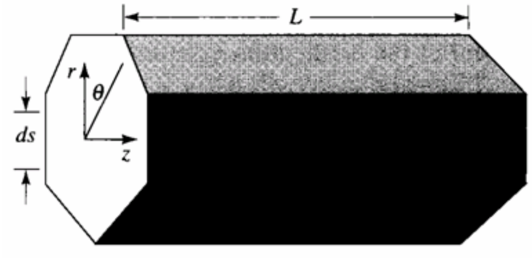
$$N = \mathrm{Re}^a\,\mathrm{Fr}^b\mathrm{We}^c \quad \left(\frac{\rho UL}{\mu}\right)^a\left(\frac{U^2}{gL}\right)^b\left(\frac{\rho U^2 L}{\sigma}\right)^c = \alpha U^0 L^0$$

$$N = \frac{\mathrm{Re}\,\mathrm{Fr}^{1/4}}{\mathrm{We}^{3/4}} = \left(\frac{\sigma^3\rho}{\mu^4 g}\right)^{1/4} \qquad \frac{\mathrm{Re}^{4/3}\,\mathrm{Fr}^{1/3}}{\mathrm{We}} = \sigma\left(\frac{\rho}{\mu^4 g}\right)^{1/3} \equiv \gamma$$

$$\boxed{\mathrm{Ca}^* = \Re(\gamma)}$$



Inspectional analysis



$$\frac{\Delta P}{\rho U^2} = \Psi[\mathrm{Re},\ \mathrm{Fr},\ \text{shape factors}]$$

$$\left(u_r^*\frac{\partial u_z^*}{\partial r^*} + \frac{u_\theta^*}{r^*}\frac{\partial u_z^*}{\partial \theta} + u_z^*\frac{\partial u_z^*}{\partial z^*}\right) = -\frac{\partial p^*}{\partial z^*} + \frac{1}{\mathrm{Re}}\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^*}{\partial r^*}\right) + \frac{\partial^2 u_z^*}{\partial z^{*2}} + \frac{1}{r^{*2}}\frac{\partial^2 u_z^*}{\partial \theta^2}\right]$$

<u>Fully developed laminar flow</u> $(u_r, u_\theta, u_z) = (0, 0, u_z(r,\theta))$

$$0 = -\frac{\partial p^*}{\partial z^*} + \frac{1}{\mathrm{Re}}\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^*}{\partial r^*}\right) + \frac{1}{r^{*2}}\frac{\partial^2 u_z^*}{\partial \theta^2}\right] \qquad 0 = -\frac{\partial p^*}{\partial r^*}$$

$$u_z^* = u_z^*(r^*, \theta, \mathrm{Re}, \text{shape factors})$$

$$p^* = p^*(z^*, \mathrm{Re}, \text{shape factors})$$

Shape factor from boundaries

$$C(z^*) = -\frac{dp^*}{dz^*} = F(r^*, \theta)$$

Constant; dimensionless pressure gradient

$$C(z^*) = -\frac{\Delta P D}{\rho U^2 L}$$

Since *C* is a constant, it depends only on the constants that appear in the differential equations and the boundary conditions that define the flow

$$\boxed{C = f'(\mathrm{Re}, \text{shape factors})}$$

$$0 = -\frac{\partial p^*}{\partial z^*} + \frac{1}{\mathrm{Re}}\left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^*}{\partial r^*}\right) + \frac{1}{r^{*2}}\frac{\partial^2 u_z^*}{\partial \theta^2}\right]$$

$$u_z^* = u_z^*(r^*, \theta, \mathrm{Re}, \text{shape factors})$$
$$p^* = p^*(z^*, \mathrm{Re}, \text{shape factors})$$

$$z^{**} = \frac{z^*}{\mathrm{Re}}$$

$$0 = -\frac{\partial p^*}{\partial z^{**}} + \left[\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^*\frac{\partial u_z^*}{\partial r^*}\right) + \frac{1}{r^{*2}}\frac{\partial^2 u_z^*}{\partial \theta^2}\right]$$

$$u_z^{**} = u_z^{**}(r^*, \theta, \text{shape factors})$$
$$p^{**} = p^{**}(z^{**}, \text{shape factors})$$

Force balance; the pressure drop arises from the wall shear stress

$$\Delta P A_c = \int_0^L \left[\underline{\int_s -\tau_{zn} ds}\right] dz$$

A contour integral along the cross-sectional perimeter

$$\frac{\Delta P}{\rho U^2} = \frac{1}{\mathrm{Re}}\int_0^{L/D}\left[\underline{\int_{s^*}} \Delta_{zn}^* ds^*\right] dz^* \quad \Rightarrow \quad \frac{\Delta P}{\rho U^2} = \frac{K'(\mathrm{Re}, \text{shape factors})}{\mathrm{Re}\, D/L}$$

$$\Delta_{zn}{}^* = f(\theta, \mathrm{Re}, \mathit{shape\ factor})$$

$$\frac{\Delta P}{\rho U^2} = \int_0^{(L/D\mathrm{Re})}\left[\int_{s^*} \Delta_{zn}^{**} ds^*\right] dz^{**} \quad \Rightarrow \quad \frac{D\Delta P}{\rho L U^2}\mathrm{Re} = K(\text{shape factors})$$

Dimensional analysis; 6-3=3 $$\frac{\Delta P}{\rho U^2} = fn(\mathrm{Re},\ L/D,\ \text{shape factors})$$

$$\Delta P A_c = \int_0^L \left[\int_s -\tau_{zn}\,ds\right]dz = \left[\int_s -\tau_{zn}\,ds\right]L = \bar{\tau} S L$$

Fully developed

Wall shear stress averaged along the perimenter; S is the wetted perimeter

Friction factor

$$f = \frac{\bar{\tau}}{\frac{1}{2}\rho U^2} = \frac{\Delta P A_c / SL}{\frac{1}{2}\rho U^2} = \frac{(\Delta P / L) r_\mathrm{h}}{\frac{1}{2}\rho U^2}$$

$$D_\mathrm{h} = 4 r_\mathrm{h} \qquad \mathrm{Re} = \frac{4 r_\mathrm{h} U \rho}{\mu}$$

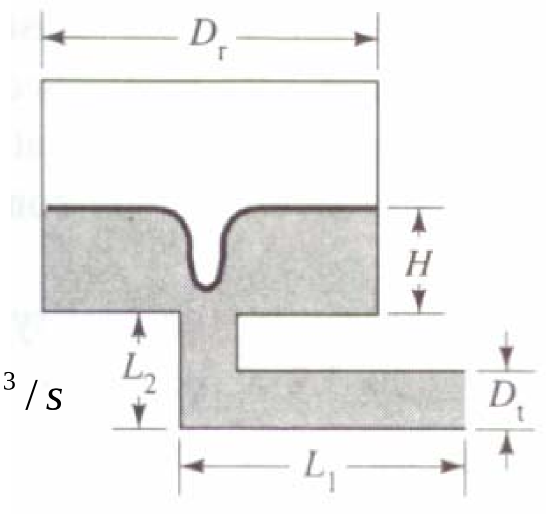
$$\boxed{f\,\mathrm{Re} = F(\text{shape factors})}$$

Entry region flow

$$f\,\mathrm{Re} = \frac{\mathrm{Re}\,D}{2L}\int_0^{(L/D\,\mathrm{Re})}\left[\int_{s^*} \Delta_{zn}^{**}\,ds^*\right]dz^{**} = \frac{\mathrm{Re}\,D}{2L}\int_0^{(L/D\,\mathrm{Re})}\left[h(z^{**})\right]dz^{**} = \frac{\mathrm{Re}\,D}{2L} G(L/D\,\mathrm{Re})$$

$$f\,\mathrm{Re} = F(\mathrm{Re}\,D/L,\ \text{shape factors})$$

Experimental design



At some critical height, the free surface forms a vortex that is sucked into the tube, entraining air in the liquid; we wish to avoid

$$D_r = 1\,\mathrm{m} \quad \underline{D_t = 0.03\,\mathrm{m}} \quad L_1 = 0.5\,\mathrm{m} \quad L_2 = 0.3\,\mathrm{m} \quad Q = 1.2\times10^{-2}\,m^3/s$$

$$\mu = 2.4\,\mathrm{Pa\cdot s} \quad \rho = 2400\,\mathrm{kg/m^3} \quad \sigma = 0.25\,\mathrm{N/m}$$

Molten ceramic at 1000K in large reservoir
-> need scale-down

Dynamic similarity

$$\mathrm{Re}_{\mathrm{model}} = \mathrm{Re}_{\mathrm{real}} \quad \mathrm{Fr}_{\mathrm{model}} = \mathrm{Fr}_{\mathrm{real}} \quad \mathrm{We}_{\mathrm{model}} = \mathrm{We}_{\mathrm{real}}$$

If the length scale is large, the radius of curvature of the vortex may be so large that surface tension effect will be negligible

$$\mathrm{Bo} = \frac{\mathrm{We}}{\mathrm{Fr}} = \frac{\rho g L^2}{\sigma} = \frac{2400\times9.8\times0.03^2}{0.25} = 84.7$$

Surface tension effect is unimportant -> neglect *We*

$$\mathrm{Re} = \frac{\rho U D_\mathrm{t}}{\mu} = \frac{4\rho Q}{\pi D_\mathrm{t}\mu} = 509 \qquad \mathrm{Fr} = \frac{U^2}{g D_\mathrm{t}} = \frac{16 Q^2}{\pi^2 g D_\mathrm{t}^5} = 980$$

3 parameters (tube diameter, flow rate, kinematic viscosity), 2 constraints
-> select tube diameter 0.3*cm* (scale down by one order of magnitude)

$$\left[\frac{Q}{\nu D_\mathrm{t}}\right]_{\mathrm{model}} = \left[\frac{Q}{\nu D_\mathrm{t}}\right]_{\mathrm{real}} \qquad \left[\frac{Q^2}{D_\mathrm{t}^5}\right]_{\mathrm{model}} = \left[\frac{Q^2}{D_\mathrm{t}^5}\right]_{\mathrm{real}}$$

$$Q_{\mathrm{model}} = 3.8\times10^{-5}\ \mathrm{m^3/s} = 38\ \mathrm{cm^3/s}$$

$$\nu_{\mathrm{model}} = 3.16\times10^{-5}\ \mathrm{m^2/s}$$

$$\Rightarrow \quad \mu_{\mathrm{model}} = 3.16\times10^{-2}\ \mathrm{Pa\cdot s}$$

Test with an aqueous solution of corn syrup or glycerol at 1/10 of full scale (geometrically and dynamically similar)

$$\mathrm{Bo} = \frac{1000(9.8)(0.003)^2}{0.06} = 1.5$$

Surface tension may be important in this small length scale

$\Rightarrow$ Need more experiments with liquids of several surface tensions and look for any influence

Priciple of dynamic similarity

- For isothermal, incompressible, low Mach number flow, the fluid dynamics are completely controlled by the values of no more than three dynamic dimensionless groups

- Reynolds number; relative importance of inertial forces to viscous force
- Froude number; of inertial force to gravitational force
- Weber number; of inertial force to interfacial force

- All the dimensionless groups that might come out from the Buckingham pi theorem can be expressible in terms of these 3 groups, hence are not independent of them