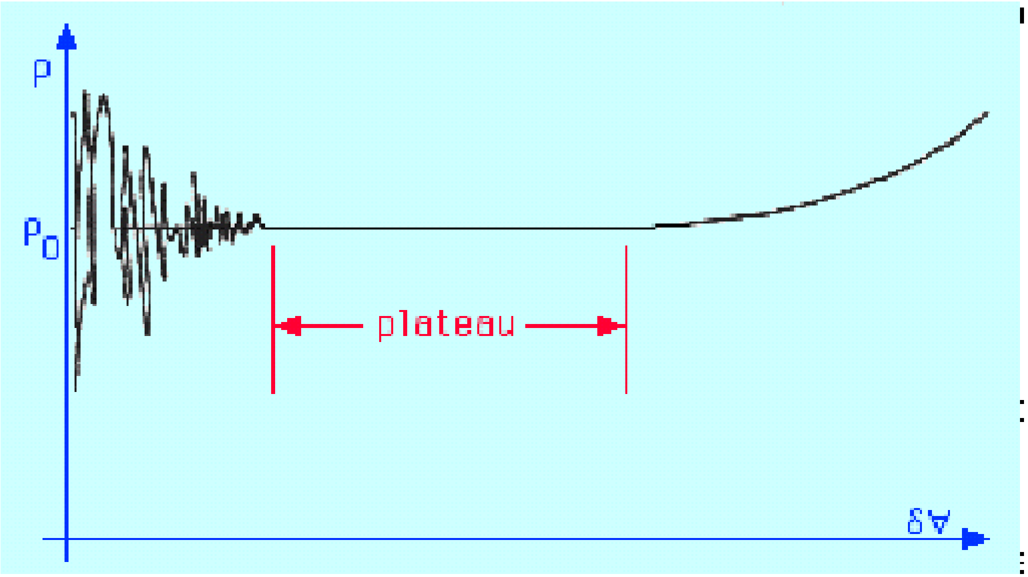
Chapter 4

Conservation of mass and momenturm

Continuum

Continuum: a region of material space throughout which properties such as velocity, temperature, density, and composition vary in a mathematically continuous manner

$$\rho \equiv \lim \frac{\Delta m}{\Delta V} \quad \text{at any point} \quad \text{as}\, \Delta V \to 0$$



Control volume

Fluid flows across the surfaces of *dV*, carrying mass in and out of the volume

Any change of mass within the volume

$$dm = \left[\left(\frac{\partial}{\partial t} \right) \overline{\rho}\, dV \right] dt$$

Mass rate of flow across a surface

= density * volume rate of flow

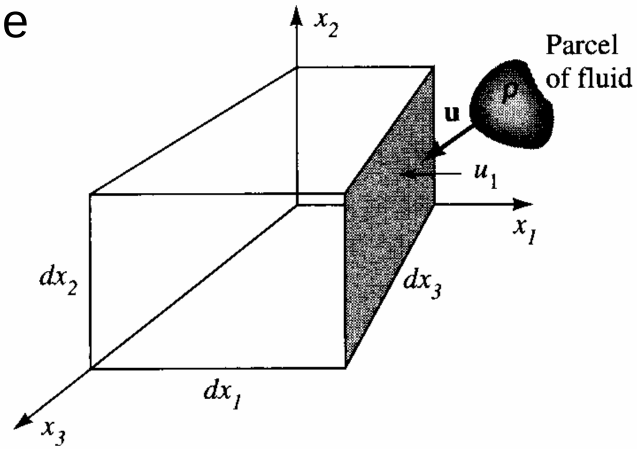


Figure 4.1.1 Volume element for derivation of the continuity equation.

$$\begin{aligned}\left(\frac{\partial}{\partial t} \right) \overline{\rho} dV = & [\rho u_1 dx_2 dx_3]_{x_1=0} - [\rho u_1 dx_2 dx_3]_{x_1=dx_1} \\ & + [\rho u_2 dx_1 dx_3]_{x_2=0} - [\rho u_2 dx_1 dx_3]_{x_2=dx_2} \\ & + [\rho u_3 dx_2 dx_1]_{x_3=0} - [\rho u_3 dx_2 dx_1]_{x_3=dx_3}\end{aligned}$$

Continuity equation

$$\frac{\partial \overline{\rho}}{\partial t} = \frac{[\rho u_1]_{x_1=0} - [\rho u_1]_{x_1=dx_1}}{dx_1} + \frac{[\rho u_2]_{x_2=0} - [\rho u_2]_{x_2=dx_2}}{dx_2} + \frac{[\rho u_3]_{x_3=0} - [\rho u_3]_{x_3=dx_3}}{dx_3}$$

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho u_1}{\partial x_1} + \frac{\partial \rho u_2}{\partial x_2} + \frac{\partial \rho u_3}{\partial x_3} \right)$$

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho u_1}{\partial x_1} + \frac{\partial \rho u_2}{\partial x_2} + \frac{\partial \rho u_3}{\partial x_3} \right)$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{u}$$

For incompressible fluids

$$\nabla \cdot \mathbf{u} = 0$$

Table 4.1.1 Components of the Divergence Operator (for any vector **A**)

Cartesian	
$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	**(4.1.8a)**
Cylindrical	
$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$	**(4.1.8b)**
Spherical	
$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta}\frac{\partial (A_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta}\frac{\partial A_\phi}{\partial \phi}$	**(4.1.8c)**

Deformation in a fluid

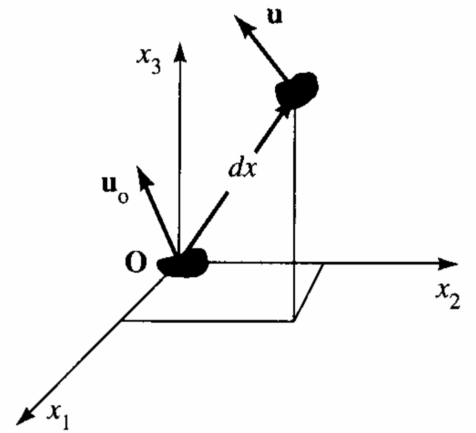


Figure 4.2.1 Deformation in the region near the point **O**.

$$u_i = u_{oi} + \left(\frac{\partial u_i}{\partial x_1}\right) dx_1 + \left(\frac{\partial u_i}{\partial x_2}\right) dx_2 + \left(\frac{\partial u_i}{\partial x_3}\right) dx_3 + \mathrm{O}(dx_i)^2$$

$$u_i - u_{oi} = \left(\frac{\partial u_i}{\partial x_1}\right) dx_1 + \left(\frac{\partial u_i}{\partial x_2}\right) dx_2 + \left(\frac{\partial u_i}{\partial x_3}\right) dx_3 = \sum_{j=1}^{3} \left(\frac{\partial u_i}{\partial x_j}\right) dx_j$$

Velocity gradient $$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial u_i}\right)$$

Rate of deformation tensor

Rigid body rotation

Summation convention

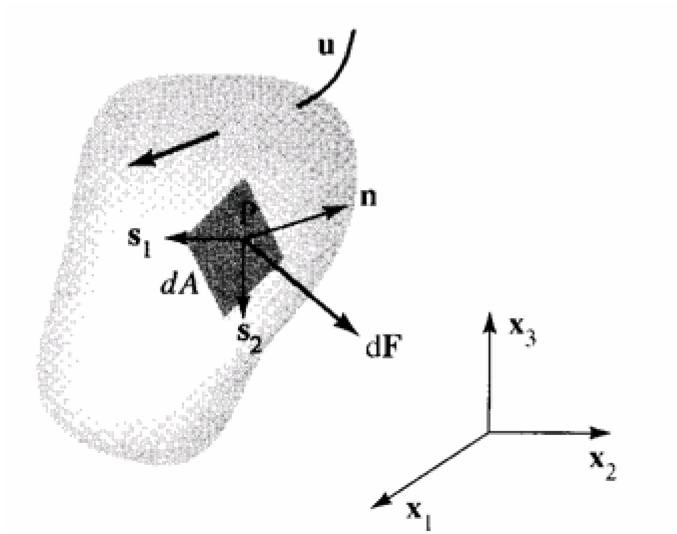
Drop the summation sign and adopt the convention of summing over any subscript that is repeated in a product

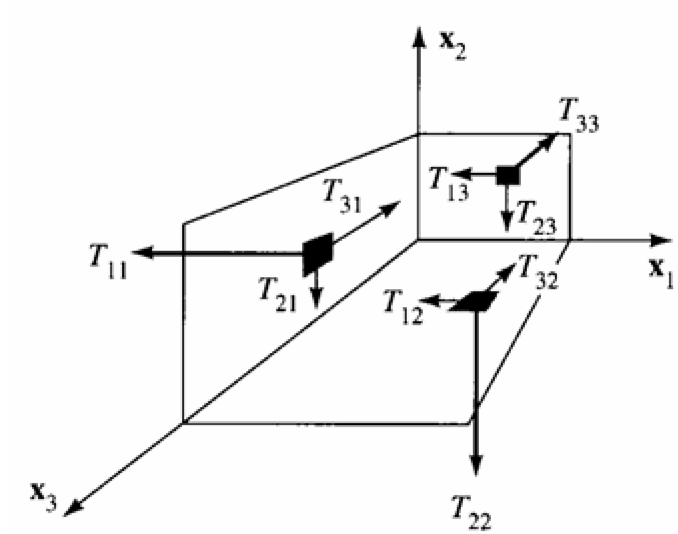
$$du_i = du_{i\Delta} + du_{i\omega} = \sum_{j=1}^{3} \left(\frac{\Delta_{ij}}{2} + \frac{\omega_{ij}}{2} \right) dx_j$$

$$du_{i\omega} = \tfrac{1}{2}\omega_{ij} dx_j$$

$$du_{i\omega} dx_i = du_{1\omega} dx_1 + du_{2\omega} dx_2 + du_{3\omega} dx_3 = 0$$

Stress in a fluid





$$\mathbf{T} = \lim \frac{d\mathbf{F}}{dA} \qquad as \qquad dA \to 0$$

Sign convention: the stress T_{ij}, due to the action of material on the positive side of the surface, acting on material on the negative side, is positive if the direction of the line of action is along negative x_i

Sum of the forces into the control volume in the x_1 direction

$$S_{1,o} = (T_{11}dx_2dx_3 + T_{12}dx_1dx_3 + T_{13}dx_1dx_2)_o$$

Sum of the forces out

$$S_{1,dx} = (T_{11}dx_2dx_3 + T_{12}dx_1dx_3 + T_{13}dx_1dx_2)_{dx}$$

Net forces in the x_1 direction

$$S_1 = (S_{1,o} - S_{1,dx_1}) = (T_{11,o} - T_{11,dx_1})dx_2dx_3 + (T_{12,o} - T_{12,dx_2})dx_1dx_3 + (T_{13,o} - T_{13,dx_3})dx_1dx_2$$

$$T_{11,o} - T_{11,dx_1} = -\frac{\partial T_{11}}{\partial x_1}dx_1$$

$$S_1 = -\frac{\partial T_{11}}{\partial x_1}dx_1dx_2dx_3 - \frac{\partial T_{12}}{\partial x_2}dx_2dx_1dx_3 - \frac{\partial T_{13}}{\partial x_3}dx_3dx_1dx_2$$

When fluid crosses the surface of the volume element, that flow carries momentum in (or out) of the volume

Rate at which momentum crosses the surface
= momentum density * volume flow rate normal to the surface

Summing the flows across the three orthogonal faces at the corner *x=0*

$$C_{1,o} = (\rho u_1)(u_1 dx_2 dx_3) + (\rho u_1)(u_2 dx_1 dx_3) + (\rho u_1)(u_3 dx_1 dx_2)$$

$$(\rho u_1 u_i)_o = (\rho u_1 u_i)_{dx_i} - \frac{\partial(\rho u_1 u_i)}{\partial x_i} dx_i \qquad \text{for} \quad i = 1, 2, \text{or } 3$$

Net rate of flow of momentum (in the x_1 direction) that crosses all six faces of the surface

$$C_1 = C_{1,0} - C_{1,dx_1} = -\frac{\partial(\rho u_1 u_1)}{\partial x_1} dx_1 dx_2 dx_3 - \frac{\partial(\rho u_1 u_2)}{\partial x_2} dx_2 dx_1 dx_3 - \frac{\partial(\rho u_1 u_3)}{dx_3} dx_3 dx_1 dx_2$$

Convective flow of momentum (in the *x1* direction)

Time rate of change of momentum $A_1 = \frac{\partial}{\partial t}(\rho u_1) dx_1 dx_2 dx_3$

Body force acting on the fluid in the x_1 direction $B_1 = (\rho f_1) dx_1 dx_2 dx_3$

time rate of change of momentum

= sum of forces + sum of convective flow of momentum

in the x_1 direction

$$A_1 = S_1 + B_1 + C_1$$

$$\frac{\partial}{\partial t}(\rho u_1) + \left[\frac{\partial(\rho u_1 u_1)}{\partial x_1} + \frac{\partial(\rho u_1 u_2)}{\partial x_2} + \frac{\partial(\rho u_1 u_3)}{dx_3} \right] = (\rho f_1) - \left[\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} \right]$$

$$\frac{\partial}{\partial t}(\rho u_i) + \left[\frac{\partial(\rho u_i u_1)}{\partial x_1} + \frac{\partial(\rho u_i u_2)}{\partial x_2} + \frac{\partial(\rho u_i u_3)}{dx_3} \right] = (\rho f_i) - \left[\frac{\partial T_{i1}}{\partial x_1} + \frac{\partial T_{i2}}{\partial x_2} + \frac{\partial T_{i3}}{\partial x_3} \right]$$

$$\rho \left[\frac{\partial}{\partial t}(u_i) + u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{dx_3} \right] = (\rho f_i) - \left[\frac{\partial T_{i1}}{\partial x_1} + \frac{\partial T_{i2}}{\partial x_2} + \frac{\partial T_{i3}}{\partial x_3} \right]$$

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right]_i = \rho \mathbf{f}_i - \left[\nabla \cdot \mathbf{T} \right]_i$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \rho\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right]_i = \rho \mathbf{f}_i - [\nabla \cdot \mathbf{T}]_i$$

4 equations; 3+9 unknowns

Constitutive equation; Newtonian fluid

$$T_{ij} = [p + \tfrac{2}{3}\mu(\nabla \cdot \mathbf{u})]\delta_{ij} - \mu\Delta_{ij} = p'\delta_{ij} - \mu\Delta_{ij}$$

General definition of the Newtonian fluid

$$\delta_{ij} = 0 \quad \text{if} \quad i \neq j$$
$$= 1 \quad \text{if} \quad i = j$$

Normal stress $\quad T_{11} = p' - \mu\Delta_{11}$ (and similarly for 22 and 33)

Shear stress $\quad T_{12} = -\mu\Delta_{12}$ (and similarly for 13, 23, *etc*.)

Stress components are symmetric $\quad T_{ij} = T_{ji}$

1+3+6 equations; 1+3+6 unknowns

Isothermal incompressible flow

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right]_i = -[\nabla p]_i + \mu[\nabla^2 \mathbf{u}]_i + \rho \mathbf{f}_i$$

The Navier-Stokes equation

Table 4.3.1 The Navier–Stokes Equations for Newtonian Isothermal Incompressible Flow

Rectangular (Cartesian) Coordinates (x, y, z)

x component

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \rho g_x \qquad \textbf{(4.3.24a)}$$

y component

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + \rho g_y \qquad \textbf{(4.3.24b)}$$

z component

$$\rho\left(\frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + \rho g_z \qquad \textbf{(4.3.24c)}$$

Cylindrical Coordinates (r, θ, z)

r component

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}\right) + \rho g_r \qquad \textbf{(4.3.24d)}$$

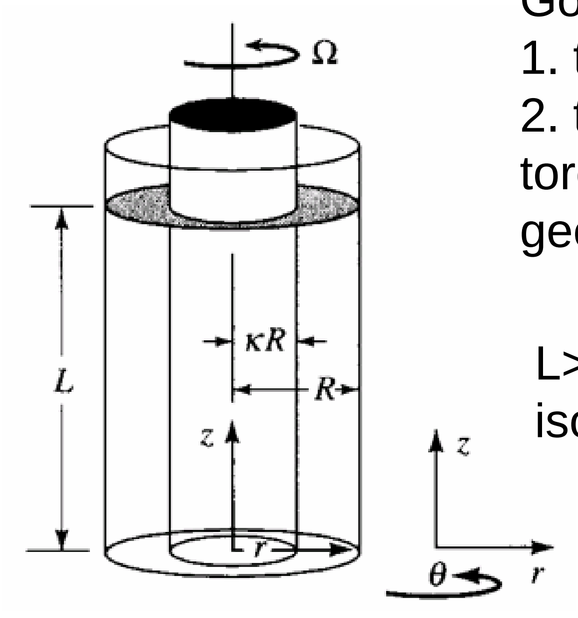
θ component

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right] + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right) + \rho g_\theta \qquad \textbf{(4.3.24e)}$$

z component

$$\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + \rho g_z \qquad \textbf{(4.3.24f)}$$

Drag flow between long concentric cylinders



Goal:
1. to find a solution for the steady state velocity vector
2. to develop a model of the relationship of the steady torque required to maintain the rotation as a function of geometry and of the liquid viscosity

L>>R
isothermal incompressible Newtonian fluid

Assumptions

1. Steady state -> $\partial / \partial t = 0$

$\partial / \partial \theta = 0; u_r = 0$

2. Concentric cylinder & laminar flow -> independent of angular position; no radial velocity

3. No motion in the *z* direction -> no flow in *z* direction, angular velocity does not vary in *z* direction, but depends only on *r*

$u_z = 0$

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right]+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right)+\rho g_r \quad \textbf{(4.3.24d)}$$

$$-\rho\frac{u_\theta^2}{r}=-\frac{\partial p}{\partial r}$$

There is a radial pressure gradient due to centrifugal acceleration

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right]+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right)+\rho g_\theta \quad \textbf{(4.3.24e)}$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right)+\rho g_z \quad \textbf{(4.3.24f)}$$

$$0=\mu\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right]$$

$$0=-\frac{\partial p}{\partial z}+\rho g_z$$

Hydrodynamic pressure gradient

$$0 = \mu \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right] \qquad \frac{1}{r}\frac{d}{dr}(r u_\theta) = a \qquad u_\theta = \frac{ar}{2} + \frac{b}{r}$$

Boundary conditions $\quad u_\theta = 0 \quad \text{at} \quad r = R$

$$u_\theta = \kappa R \Omega \quad \text{at} \quad r = \kappa R$$

Velocity field

$$\boxed{u_\theta = \frac{\kappa R \Omega}{\kappa - 1/\kappa}\left(\frac{r}{R} - \frac{R}{r}\right)}$$

Shear force acts in the θ direction on the surface normal to the *r* direction

Shear stress

$$T_{\theta r} = -\mu\left[r \frac{d}{dr}\left(\frac{u_\theta}{r}\right)\right] = \frac{2\mu\Omega}{1-\kappa^2} \quad \text{at} \quad r = \kappa R$$

Torque on the inner cylinder = tangential force * moment of arm of radius *kR*

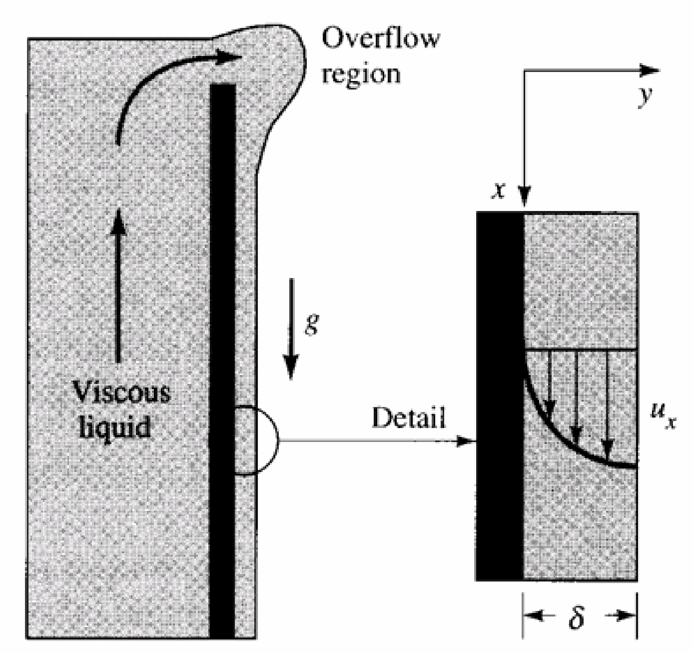
$$\mathbf{T} = \left(\frac{2\mu\Omega}{1-\kappa^2}\right)(2\pi\kappa R L)(\kappa R) = \frac{4\pi\mu L \kappa^2 R^2 \Omega}{1-\kappa^2}$$

Coaxial cylinder viscometer

Validity of assumptions

- Steady state $$t_\infty = \frac{\rho R^2}{\mu}$$
- Newtonian fluid
- Very high viscosity liquid -> temperature rise due to viscous heating
- Effect of bottom face
- Laminar flow $$\frac{\Omega R^2 \rho}{\mu} < \frac{40}{\kappa(1-\kappa)^{3/2}}$$

Liquid film on a vertical surface



Goal: to discover a relationship between the liquid film thickness and the flow rate

Assumptions

1. Steady state
2. Isothermal, incompressible, Newtonian
3. laminar, strictly parallel to the plate -> no flow in *y* and z direction, no *x* derivatives, the only velocity is u_x
4. fully developed (?) -> film thickness is not a function of *x*, but only of *y*

Continuity equation $$0 = \nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x}$$

All from the assumption

$$u_y = u_z = 0$$

$$Q = W \int_0^{\delta} u_x dy \qquad \frac{1}{W}\frac{dQ}{dx} = 0 = \int_0^{\delta} \frac{\partial u_x}{\partial x} dy + u_x(\delta)\frac{d\delta}{dx}$$

Leibniz formula $$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + f(b,t)\frac{db}{dt} - f(a,t)\frac{da}{dt}$$

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \rho g_x \qquad \textbf{(4.3.24a)}$$

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + \rho g_y \qquad \textbf{(4.3.24b)}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + \rho g_z \qquad \textbf{(4.3.24c)}$$

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial y^2}\right) + \rho g_x \qquad 0 = -\frac{\partial p}{\partial y} \qquad 0 = 0$$

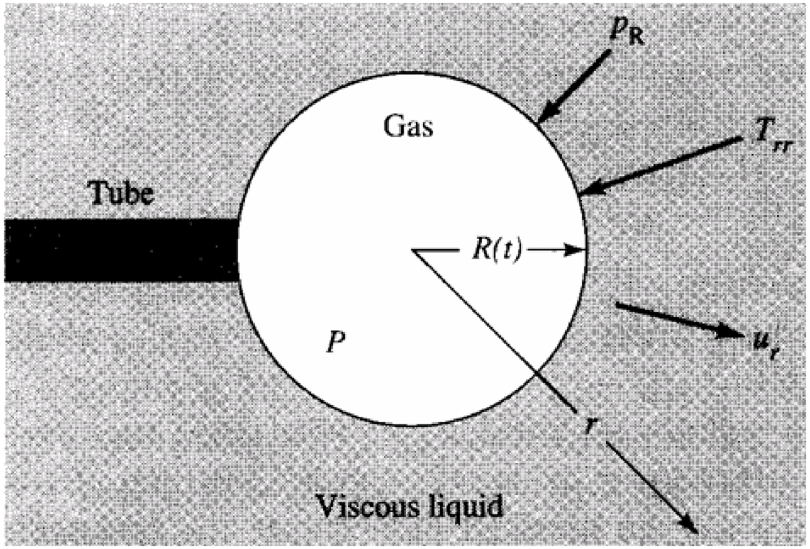
No pressure gradient at the free surface

$$u_x = -\frac{\rho g_x}{2\mu} y^2 + ay + b$$

BC $u_x = 0$ at $y = 0$ $\quad T_{xy} = -\mu\frac{du_x}{dy} = 0$ at $y = \delta$

$$\boxed{u_x = \frac{\rho g_x \delta^2}{2\mu}\left[\frac{2y}{\delta} - \left(\frac{y}{\delta}\right)^2\right]} \qquad Q = \frac{\rho g_x W \delta^3}{3\mu} \qquad \mathrm{Re} = \frac{4\rho\delta\langle u_x\rangle}{\mu} = \frac{4\rho Q}{\mu W} < 10$$

Growth of a bubble in a viscous liquid



Goal: to develop a model for the relationship of the rate of growth of the bubble to the pressure and to the properties of the fluid

Assumptions

1. incompressible, isothermal, Newtonian
2. bubble remains a sphere as it grows -> the only velocity component is radial, a function only of *r* and *t*
3. gravity is neglected

Continuity in spherical coordinates

$$\frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(A_\theta \sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \qquad \textbf{(4.1.8c)}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = 0 \qquad u_r = \frac{A(t)}{r^2} \qquad u_r\big|_{r=R} = \frac{dR}{dt} = \dot{R} = \frac{A}{R^2} \qquad u_r = \frac{R^2\dot{R}}{r^2}$$

Navier-Stokes equation

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + \frac{u_\phi}{r\sin\theta}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi^2 + u_\theta^2}{r}\right)$$

$$= -\frac{\partial p}{\partial r} + \mu\left(\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 u_r)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_r}{\partial\phi^2}\right) + \rho g_r$$

Young-Laplace equation $\quad P = T_{rr}\big|_R + \dfrac{2\sigma}{R}$

Newtonian fluid $\quad T_{rr} = p - \mu\Delta_{rr} = 2\dfrac{\partial u_r}{\partial r} \quad$ Extensional flow

$$P(t) = \frac{2\sigma}{R} + \frac{4\mu\dot{R}}{R} + p_\infty + \rho\left(R\ddot{R} + \frac{3\dot{R}^2}{2}\right)$$

Growth of a bubble in a very viscous liquid

Wire coating of a highly viscous molten polymer

Bubble, pressure 10^6Pa (maintained), initial radius 100μm, final radius 1mm
Medium, viscosity 100Pa.s

If we neglect the contribution of surface tension and inertia

$$P(t) = \frac{2\sigma}{R} + \frac{4\mu\dot{R}}{R} + p_\infty + \rho\left(R\ddot{R} + \frac{3\dot{R}^2}{2}\right) \qquad \frac{4\mu\dot{R}}{R} = P - p_\infty = \Delta P \qquad \frac{R}{R_o} = \exp\left(\frac{\Delta P t}{4\mu}\right)$$

$$\ln\frac{R}{R_o} = \left(\frac{\Delta P t}{4\mu}\right) = \ln 10 = 2.3 \qquad t = \frac{9.2\mu}{\Delta P} = \frac{9.2\times 100}{9\times 10^5} \approx 10^{-3}\ \mathrm{s}$$

Contribution of surface tension $\quad P_\sigma = \frac{2\sigma}{R} \approx \frac{2\times 0.05\ \mathrm{N/m}}{10^{-4}\ \mathrm{m}} = 10^3\ \mathrm{N/m^2} = 10^3\ \mathrm{Pa}$

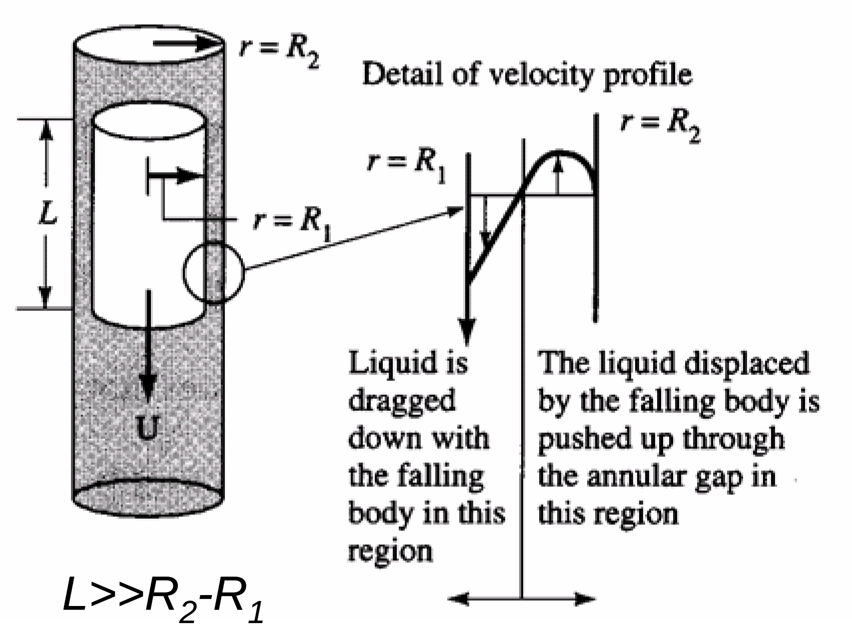
Inertial effect

$$\frac{\mathrm{I}}{\mathrm{V}} \equiv \frac{5}{32}\frac{\rho\Delta P R_o^2}{\mu^2}\left(\frac{R}{R_o}\right)^2$$

$$\frac{\mathrm{I}}{\mathrm{V}} = \frac{5}{32}\frac{10^3\ \mathrm{kg/m^3}\times 9\times 10^5\ \mathrm{Pa}\times 10^{-8}\ \mathrm{m^2}}{(100)^2\ \mathrm{Pa^2}\cdot \mathrm{s^2}}\left(\frac{R}{R_o}\right)^2 \approx 1.4\times 10^{-4}\left(\frac{R}{R_o}\right)^2$$

In real systems, the internal pressure may not be constant

Viscous resistance to a freely falling objects



$L >> R_2 - R_1$

Moving coordinate system;
z=0 at the upper surface of the cylinder

Assumptions

1. Steady state
2. incompressible, isothermal, Newtonian
3. fully developed flow -> ignores end effects, laminar flow, axi-symmetric

Continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u_\theta) + \frac{\partial}{\partial z}(\rho u_z) = 0$$

$$r u_r = \text{constant for all } r \qquad r u_r = 0 \quad \text{on} \quad r = R_1 \quad \text{and} \quad R_2$$

$u_r = 0$ everywhere, consequence of the assumptions of axisymmetric, constant density, fully developed flow

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right]+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right)+\rho g_r \qquad \text{(4.3.24d)}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right]+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right)+\rho g_\theta \qquad \text{(4.3.24e)}$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right)+\rho g_z \qquad \text{(4.3.24f)}$$

$$\left.\begin{aligned}u_r&=0\\u_\theta&=0\\g_r&=0\end{aligned}\right\}\text{ leads to }\frac{\partial p}{\partial r}=0$$

$$\left.\begin{aligned}u_\theta&=0\\\frac{\partial}{\partial\theta}&=0\\g_\theta&=0\end{aligned}\right\}\text{ leads to }\frac{\partial p}{\partial \theta}=0$$

$$p = p(z)\text{ only}$$

$$\left.\begin{aligned}\frac{\partial}{\partial t}&=0\\u_r&=0\\u_\theta&=0\\\frac{\partial u_z}{\partial z}&=0\\\frac{\partial}{\partial\theta}&=0\end{aligned}\right\}\text{ leads to }0=-\frac{\partial p}{\partial z}+\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\rho g_z$$

f'n of *z* — f'n of *r* — Const.

$$0=F(z)+G(r) \qquad F(z)=\frac{\partial p}{\partial z}-\rho g_z=-G(r)$$

$$F=-G=\text{constant}=C$$

$$\frac{dp}{dz} - \rho g_z = C \quad \text{and} \quad \mu \frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr} \right) = C$$

$$p - p_o = (C + \rho g_z)z \qquad p - \rho g z = P \qquad P - P_o = Cz \qquad C = \frac{P_L - P_o}{L} = -\frac{\Delta P}{L}$$

$$\mu \frac{1}{r}\frac{d}{dr}\left(r\frac{du_z}{dr} \right) = -\frac{\Delta P}{L} \qquad \boxed{u_z = -\frac{\Delta P}{4\mu L} r^2 + A\ln r + B}$$

No-slip boundary conditions

$$u_z = -U \quad \text{on} \quad r = R_2 \qquad u_z = 0 \quad \text{on} \quad r = R_1 = \kappa R_2$$

Non-dimensionalize

$$\frac{u_z}{U} = \phi \qquad \frac{\Delta P R_2^2}{4\mu L U} = \Phi \qquad \frac{r}{R_2} = s \qquad \phi = -\Phi s^2 + \frac{A}{U}\ln sR_2 + \frac{B}{U}$$

$$\boxed{\phi(s) = -1 + \Phi(1 - s^2) + \frac{1 - \Phi(1-\kappa^2)}{\ln\kappa}\ln s}$$

ΔP is unknown, *U* is unkown
-> need two more conditions

1.The liquid displaced by the falling cylinder flow through the annular region

$$\int_{R_1}^{R_2} 2\pi u_z'(r) r\, dr = -\pi R_1^2 U = -\pi \kappa^2 R_2^2 U$$

$$\Phi = \frac{1}{(1+\kappa^2)\ln\kappa + (1-\kappa^2)}$$

2.The stresses acting on the surface of the falling cylinder retard the fall of the cylinder under the influence of gravity

Shear force + pressure force = gravity force

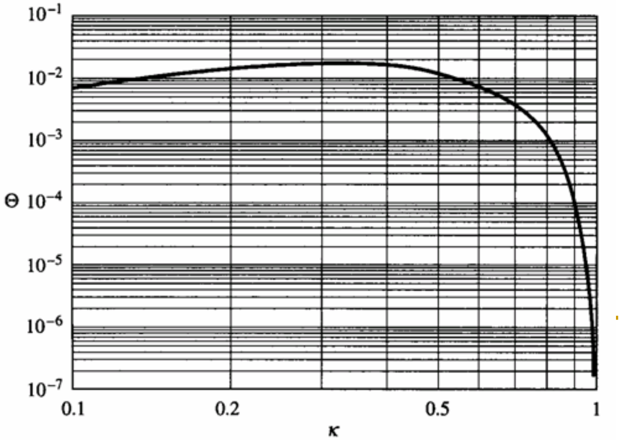
$$+\tau_{rz} A_1 + (p_L - p_o) A_2 = \text{body force} = \rho_{solid} g L A_2$$

$$\tau_{rz} A_1 + (p_L - p_o) A_2 - \rho_{fluid} g L A_2 = \text{net body force}$$

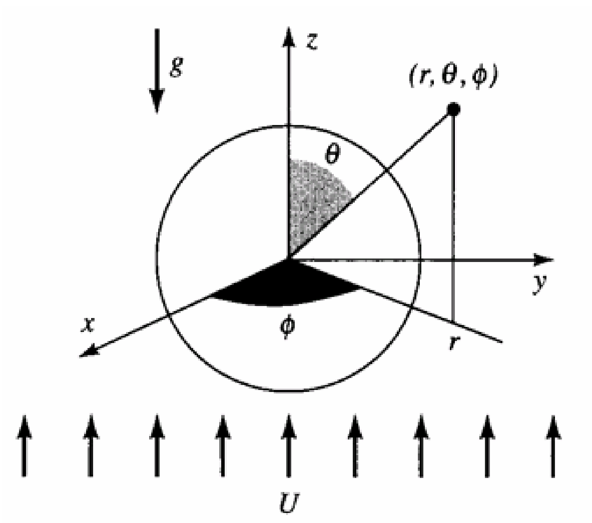
$$= \rho_{solid} g L A_2 - \rho_{fluid} g L A_2$$

$$\tau_{rz} A_1 - \Delta P A_2 = \Delta\rho g L A_2$$

$$\Theta \equiv \frac{\mu U}{\Delta\rho g R_2^2} = -\frac{\kappa^2}{2J} = \frac{\kappa^2}{2}\left[\ln\frac{1}{\kappa} - \frac{(1-\kappa^2)}{(1+\kappa^2)}\right]$$



Slow flow around a solid sphere



The flow is laminar and symmetric about z-axis

$$\partial/\partial\phi = 0, u_\phi = 0$$

$$\rho\left[u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r}\right]=-\frac{\partial p}{\partial r}+\mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2u_r)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_r}{\partial\theta}\right)\right]$$

$$\rho\left[u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}\right]=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right)\right.$$
$$\left.+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta)\right)+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}\right]$$

$$0=\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u_\theta)\right]$$

$$s\equiv\frac{r}{R},\quad \tilde{u}_r\equiv\frac{u_r}{U},\quad \tilde{u}_\theta\equiv\frac{u_\theta}{U},\quad \tilde{p}=\frac{p}{\mu U/R}$$

$$\frac{\rho UR}{\mu}\left[\tilde{u}_r\frac{\partial\tilde{u}_r}{\partial s}+\frac{\tilde{u}_\theta}{s}\frac{\partial\tilde{u}_r}{\partial\theta}-\frac{\tilde{u}_\theta^2}{s}\right]=-\frac{\partial\tilde{p}}{\partial s}+\left[\frac{1}{s^2}\frac{\partial^2}{\partial s^2}(s^2\tilde{u}_r)+\frac{1}{s^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{u}_r}{\partial\theta}\right)\right]$$

$$\frac{\rho UR}{\mu}\left[\tilde{u}_r\frac{\partial\tilde{u}_\theta}{\partial s}+\frac{\tilde{u}_\theta}{s}\frac{\partial\tilde{u}_\theta}{\partial\theta}+\frac{\tilde{u}_r\tilde{u}_\theta}{s}\right]=-\frac{1}{s}\frac{\partial\tilde{p}}{\partial\theta}+\left[\frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\frac{\partial\tilde{u}_\theta}{\partial s}\right)\right.$$
$$\left.+\frac{1}{s^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\tilde{u}_\theta\sin\theta)\right)+\frac{2}{s^2}\frac{\partial\tilde{u}_r}{\partial\theta}\right]$$

Creeping flow

Stokes flow; *Re=0*

BCs

$$\tilde{u}_\theta = \tilde{u}_r = 0 \quad \text{on the surface} \quad r = R$$

$$\left.\begin{aligned} u_\theta &= -U\sin\theta \\ u_r &= U\cos\theta \end{aligned}\right\} \quad \text{for} \quad r \to \infty$$

⇒ $\tilde{u}_\theta = G(s)\sin\theta$ and $\tilde{u}_r = F(s)\cos\theta$

Continuity $\left[F(s) + \frac{s}{2}\frac{dF(s)}{ds} \right] + G(s) = 0$

Cross differentiation of NS equations, and continutiy

$$s^4 \frac{d^4F(s)}{ds^4} + 8s^3\frac{d^3F(s)}{ds^3} + 8s^2\frac{d^2F(s)}{ds^2} - 8s\frac{dF(s)}{ds} = 0$$

Linear homogeneous ode

$$F(s) = s^n \qquad F(s) = as^{-3} + bs^{-1} + cs^0 + ds^2$$

$$F = 0 \quad \text{on} \quad s = 1 \qquad G = 0 \quad \text{on} \quad s = 1$$

$$F = 1 \quad \text{for} \quad s \to \infty \qquad G = -1 \quad \text{for} \quad s \to \infty$$

⇒

$$\frac{dF}{ds} = 0 \quad \text{on} \quad s = 1$$

$$\frac{dF}{ds} = 0 \quad \text{for} \quad s \to \infty$$

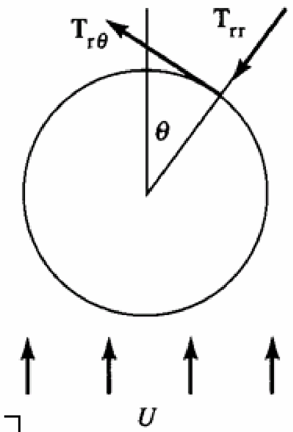
$$F(s) = 1 - \frac{3}{2s} + \frac{1}{2s^3}$$

$$G(s) = -1 + \frac{3}{4s} + \frac{1}{4s^3}$$

Velocity profile *(s=r/R)*

$$\tilde{p} = \tilde{p}_\infty - \frac{3}{2s^2}\cos\theta$$

The stress acting on the surface of the sphere



Force due to the normal stress in the direction of motion

$$F_P = \int_0^{2\pi}\int_0^{\pi}[-T_{rr}(R,\theta)\cos\theta]R^2\sin\theta\, d\theta\, d\phi = 2\pi\int_0^{\pi}\left[\left(-p+2\mu\frac{\partial u_r}{\partial r}\right)_R\cos\theta\right]R^2\sin\theta\, d\theta$$

$$F_P = 2\pi\mu RU$$

Force due to the shear stress in the direction of motion

$$F_V = \int_0^{2\pi}\int_0^{\pi}[T_{r\theta}(R,\theta)\sin\theta]R^2\sin\theta\, d\theta\, d\phi$$

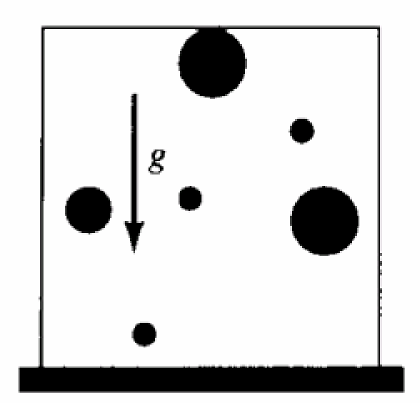
$$F_V = 4\pi\mu RU$$

Total drag force $\quad F_{SL} = F_P + F_V = 6\pi\mu RU$

Form drag (F_P) | Frictional drag (F_V)

$$\mathrm{Re} = \frac{2\rho_f UR}{\mu} \le 0.1 \quad \text{for Stokes' law to hold}$$

Sedimentation of small particles



Gravitational force = weight - buoyancy

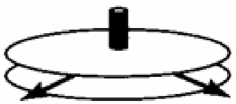
$$F_g = \tfrac{4}{3}\pi R^3(\rho_s - \rho_f)\,g = 6\pi\mu R U$$

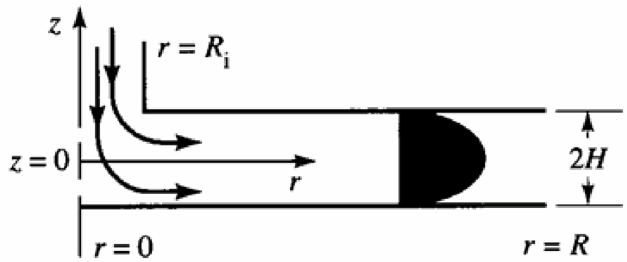
$$U = \frac{2R^2}{9\mu}(\rho_s - \rho_f)\,g$$

U proportional to R^2 -> used to separate small particles

Stokes's law holds, provided that the spheres are no closer than 15 radii to one another, Reynolds number is small, Newtonian…

Radial flow between parallel disks





Goal: find a relationship between the flowrate and the pressure forcing the flow

Assumptions

1. The disks are rigid, no-slip, surfaces
2. The flow is symmetric about the z axis -> $\partial / \partial \theta = 0, \; u_\theta = 0$
3. Newtonian, isothermal, incompressible
4. The flow is steady in time

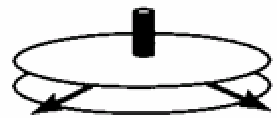
Continuity $$0 = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_z}{\partial z}$$

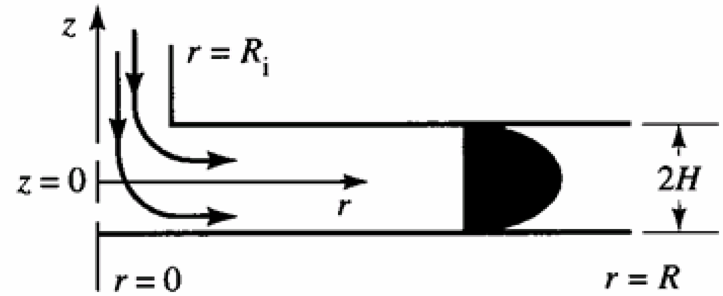
$$\rho\left(\cancel{\frac{\partial u_r}{\partial t}} + u_r\frac{\partial u_r}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}} - \cancel{\frac{u_\theta^2}{r}} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right] + \cancel{\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial^2 u_r}{\partial z^2}\right) + \cancel{\rho g_r} \quad \textbf{(4.3.24d)}$$

$$\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right] + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right) + \rho g_\theta \quad \textbf{(4.3.24e)}$$

$$\rho\left(\cancel{\frac{\partial u_z}{\partial t}} + u_r\frac{\partial u_z}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}} + \frac{\partial^2 u_z}{\partial z^2}\right) + \underbrace{\rho g_z}_{-\rho g} \quad \textbf{(4.3.24f)}$$

Boundary conditions





$$u_r = u_z = 0 \quad \text{on} \quad z = -H,\ 0 \le r \le R$$

$$u_r = u_z = 0 \quad \text{on} \quad z = +H,\ R_\mathrm{i} \le r \le R$$

$$u_r = u_z = 0 \quad \text{on} \quad r = R_\mathrm{i},\ H \le z \le H + L_\mathrm{e}$$

$$u_r = 0 \quad \text{and} \quad \frac{\partial u_z}{\partial r} = 0 \quad \text{on} \quad r = 0,\ -H \le z \le H + L_\mathrm{e}$$

$$p = 0 \quad \text{for} \quad -H \le z \le +H,\ r = R$$

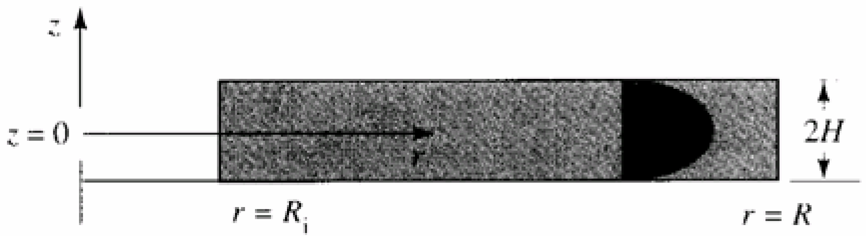
either

$$p = P_\mathrm{entr} \quad \text{at} \quad z = H + L_\mathrm{e},\ 0 \le r \le R_\mathrm{i}$$

$$u_z = -U \quad \text{and} \quad u_r = 0 \quad \text{at} \quad z = H + L_\mathrm{e},\ 0 \le r \le R_\mathrm{i}$$

More assumptions:

1. Neglect the entry tube
2. Velocity vector is strictly parallel to the plane $\mathbf{u} = [u_r(r,z), 0, 0]$



$$0 = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \cancel{\frac{\partial u_z}{\partial z}}$$

$$\rho\left(\cancel{\frac{\partial u_r}{\partial t}} + u_r\frac{\partial u_r}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}} - \cancel{\frac{u_\theta^2}{r}} + \cancel{u_z\frac{\partial u_r}{\partial z}}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right] + \cancel{\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial^2 u_r}{\partial z^2}\right) + \cancel{\rho g_r} \quad \textbf{(4.3.24d)}$$

continuity

$$\rho u_r \frac{\partial u_r}{\partial r} = -\frac{\partial p}{\partial r} + \mu\frac{\partial^2 u_r}{\partial z^2}$$

nondimensionalize $\quad \frac{u_r}{U} = u^*, \quad \frac{p}{\mu U/H} = p^*, \quad \frac{r}{H} = r^*, \quad \frac{z}{H} = z^*$

$$\left(\cancel{\frac{\rho U H}{\mu}}\right)u^*\frac{\partial u^*}{\partial r^*} = -\frac{\partial p^*}{\partial r^*} + \frac{\partial^2 u^*}{\partial {z^*}^2}$$

Creeping flow

Continuity $0 = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z}$ $u^* = \frac{1}{r^*}C(z^*)$ $0 = -\frac{\partial p^*}{\partial r^*} + \frac{1}{r^*}\frac{\partial^2 C}{\partial z^{*2}}$

$$r^*\frac{\partial p^*}{\partial r^*} = \frac{\partial^2 C}{\partial z^{*2}} = A$$

$$\rho\left[u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right] = -\frac{\partial p}{\partial z} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u_z}{\partial r}\right] + \frac{\partial^2 u_z}{\partial z^2}\right\} - \rho g$$

$\mathbf{u} = [u_r(r,z), 0, 0]$

$p = -\rho g z + fn(r)$ $\partial p / \partial r$ is not a function of z

$C = -\frac{A}{2}[1-(z^*)^2]$ (use no-slip BC) $p^* = A\ln r^* + P^* - \frac{\rho g}{\mu U}z^*$

Hydrostatic pressure affects little on the flow

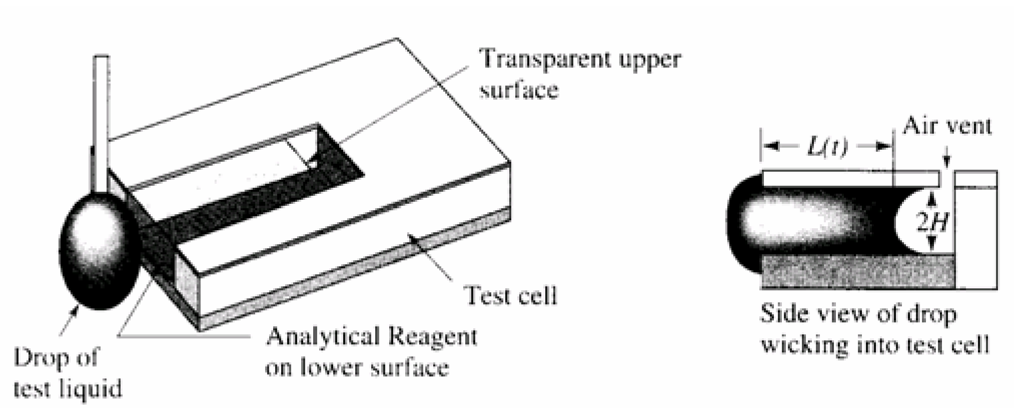
$p^* = 0$ at $r^* = \frac{R}{H}$ $P^* = -A\ln\frac{R}{H}$

$Q = 2\int_0^H 2\pi(ru_r)\,dz = 2H^2U\int_0^1 2\pi C\,dz^*$ $Q = 4\pi R_i HU$ $A = -\frac{3R_i}{H}$

$u_r(r,z) = \left(\frac{3Q}{8\pi H}\right)\frac{1}{r}\left[1-\left(\frac{z}{H}\right)^2\right]$

$$\Delta P = -\frac{3\mu Q}{4\pi H^3}\ln\frac{R_i}{R}$$

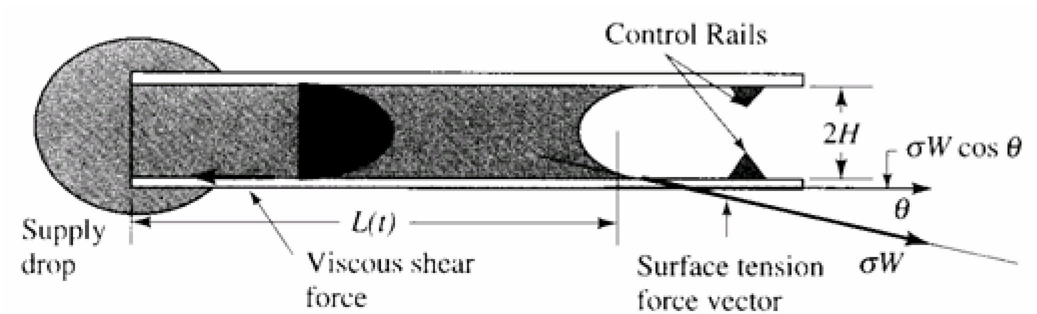
Biomedical flow device



Issue: the rate at which the sample liquid is drawn into the test cell

Driving force = surface tension

$$\frac{F_\sigma}{W} = 2\sigma\cos\theta$$



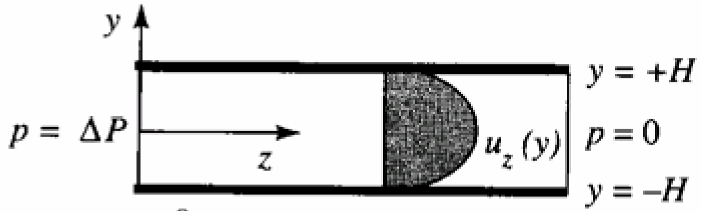
The flow is retarded by the viscous resistance of the liquid to deformation

Transient flow with free surfaces

Assumptions:

1. fully developed steady flow
2. the channel is completely filled with the liquid
3. W>>H, L>>H

$$\mathbf{u} = (u_z(y), 0, 0)$$



$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \rho g_x \qquad \textbf{(4.3.24a)}$$

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + \rho g_y \qquad \textbf{(4.3.24b)}$$

$$\rho\left(\frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + \rho g_z \qquad \textbf{(4.3.24c)}$$

$$\frac{\partial p}{\partial z} = \mu\frac{\partial^2 u_z}{\partial y^2} = C \qquad u_z = -\frac{CH^2}{2\mu}\left[1 - \frac{y^2}{H^2}\right] \qquad p = \Delta P + Cz$$

$$\frac{Q}{W} = 2\int_0^H u_z\, dy = -\frac{2CH^3}{3\mu} \qquad -C = \frac{\Delta P}{L} = \frac{3\mu Q}{2WH^3} \qquad \boxed{Q = \frac{2WH^3\Delta P}{3\mu L}}$$

Poisseuille's law

Based on the idea that the channel is completely wetted with liquid over the entire length L

Assumption: if a liquid is moving into a parallel plate channel under a constant pressure differential, Poiseuille's law holds in this transient case and we simply replace *L* by *L(t)*

Relate the pressure drop driving the flow to the surface tension

Young-Laplace equation

$$p_i - p_o = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \qquad R = \frac{H}{\cos\theta}$$

A plane corresponds to a sphere of infinite radius

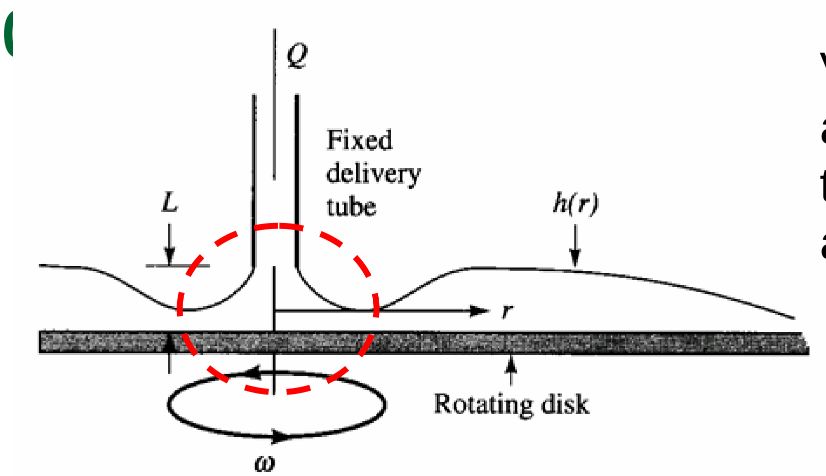
$$\Delta P = \frac{\sigma\cos\theta}{H} \qquad Q = \frac{2WH^2\sigma\cos\theta}{3\mu L(t)} \qquad Q = 2HW\frac{dL}{dt}$$

$$L = \left[\frac{2H\sigma\cos\theta}{3\mu}t\right]^{1/2} \qquad t^* = \left(\frac{3\mu}{2H\sigma\cos\theta}\right)(L^*)^2$$

*t**, *L**; required time and extent of penetration

$$\mu = 0.003\,Pa\cdot s \quad \sigma = 0.05\,N/m \quad \theta = 85^\circ \quad H = 0.0005\,m \quad L^* = 0.03\,m \quad t^* = 1.9\,s$$

Film thickness on a center-fed rotating



Very complex flow field near tube exit, but assume that beyond some radial position the film profile is smooth in some sense, and we seek a model for *h(r)* in that region

Continuity $$0 = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \cancel{\frac{\partial u_z}{\partial z}}$$

Assume the flow is largely radial, with only a small u_z component $$r u_r \neq f(r)$$

$$\rho\left(\cancel{\frac{\partial u_r}{\partial t}} + u_r\frac{\partial u_r}{\partial r} + \cancel{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}} - \frac{u_\theta^2}{r} + \cancel{u_z\frac{\partial u_r}{\partial z}}\right) = -\frac{\partial p}{\partial r} + \mu\left(\cancel{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\right]} + \cancel{\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}} + \frac{\partial^2 u_r}{\partial z^2}\right) + \cancel{\rho g_r} \quad \text{(4.3.24d)}$$

Assume the angular velocity of the liquid matches that of the disk $$u_\theta = r\omega$$

$$\rho u_r \frac{\partial u_r}{\partial r} - \rho r \omega^2 = -\frac{\partial p}{\partial r} + \mu\frac{\partial^2 u_r}{\partial z^2}$$

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}+u_z\frac{\partial u_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right)+\rho g_z \quad \textbf{(4.3.24f)}$$

For thin film, the contribution of gravity to the pressure distribution is small $\frac{\partial p}{\partial z}\approx 0$

p is uniform across the thickness of the film

Assume that the shape of the free surface is smooth, then external pressure is uniform $\frac{\partial p}{\partial r}\approx 0$

$$\rho u_r\frac{\partial u_r}{\partial r}-\rho r\omega^2=\mu\frac{\partial^2 u_r}{\partial z^2}$$

Reasonable assumptions: steady state, axisymmetric, rigid motion in angular direction

Less obvious assumptions: thin and fairly uniform thickness

Just remove the nonlinear term, and check the validity later

$$-\rho r\omega^2 = \mu \frac{\partial^2 u_r}{\partial z^2}$$

$$\frac{\partial u_r}{\partial z} = 0 \quad \text{on} \quad z = h(r)$$

No shear stress on the free surface

$$u_r = 0 \quad \text{on} \quad z = 0$$

No-slip boundary condition

$$r u_r \neq f(r)$$

$$u_r = \frac{\rho r\omega^2 h^2}{\mu}\left[\frac{z}{h} - \frac{1}{2}\left(\frac{z}{h}\right)^2\right]$$

Does not satisfy the continuity equation; we have some uncertainty as to the value of this solution, but practically quite valid

$$Q \equiv 2\pi r\int_0^{h(r)} u_r dz = \frac{2\pi\rho r^2\omega^2 h^3}{3\mu}$$

$$\boxed{h(r) = \left(\frac{3\mu Q}{2\pi\rho r^2\omega^2}\right)^{1/3}}$$

Film thickness distribution

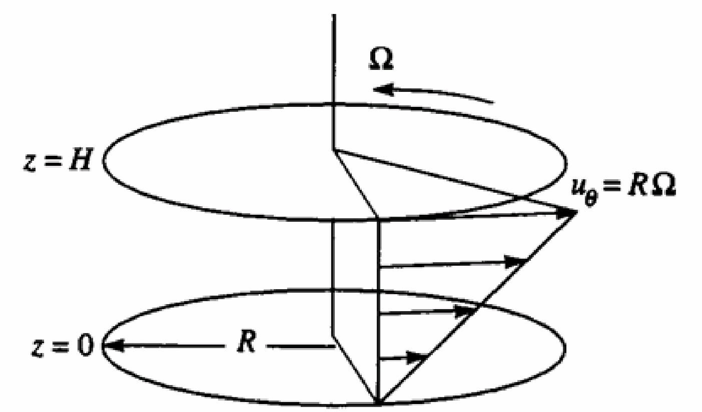
$$\overline{H} \equiv \frac{1}{\pi R^2}\int_0^R 2\pi r h(r) dr = \left(\frac{81\mu Q}{16\pi\rho R^2\omega^2}\right)^{1/3} \qquad \frac{h(r)}{\overline{H}} = \frac{2}{3}\left(\frac{r}{R}\right)^{-2/3}$$

$$\sigma \equiv \left\{\frac{1}{\pi R^2}\int_0^R 2\pi r\left[\frac{h(r)-\overline{H}}{\overline{H}}\right]^2 dr\right\}^{1/2} = 0.5774$$

$$\sigma_{1/2} \equiv \left\{\frac{4}{3\pi R^2}\int_{R/2}^R 2\pi r\left[\frac{h(r)-\overline{H}}{\overline{H}}\right]^2 dr\right\}^{1/2} = 0.2232$$

1. The degree of nonuniformity is independent of parameters -> no control over uniformity in this flow
2. Thickness is very nonuniform

Torsional flow between two parallel disks



Assumptions: laminar, Newtonian, incompressible, isothermal, axisymmetric flow at rotational speeds low enough to prevent radial flow

$$\rho\left(\cancel{\frac{\partial u_\theta}{\partial t}} + \cancel{u_r\frac{\partial u_\theta}{\partial r}} + \cancel{\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}} + \cancel{\frac{u_r u_\theta}{r}} + \cancel{u_z\frac{\partial u_\theta}{\partial z}}\right) = -\cancel{\frac{1}{r}\frac{\partial p}{\partial \theta}} + \mu\left(\underline{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)\right]} + \cancel{\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2}} + \cancel{\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}} + \underline{\frac{\partial^2 u_\theta}{\partial z^2}}\right) + \cancel{\rho g_\theta} \quad \textbf{(4.3.24e)}$$

Boundary conditions

$$u_\theta = 0 \quad \text{on} \quad z = 0$$
$$u_\theta = r\Omega \quad \text{on} \quad z = H$$

$$u_\theta = r\Omega F(z)$$

$$F(z) = 0 \quad \text{on} \quad z = 0$$
$$F(z) = 1 \quad \text{on} \quad z = H$$

Simplest $F(z) = \frac{z}{H}$

$$\boxed{u_\theta = \frac{\mathrm{rz}\Omega}{H}}$$

$$T_{\theta z} = -\mu\Delta_{\theta z} = -\mu\frac{\partial u_\theta}{\partial z} = -\frac{r\Omega\mu}{H}$$

$$\boxed{\mathrm{T} = \int_0^R -rT_{\theta z}\,dA = \int_0^R \frac{r^2\Omega\mu}{H}2\pi r\,dr = \frac{\pi\Omega\mu R^4}{2H}}$$