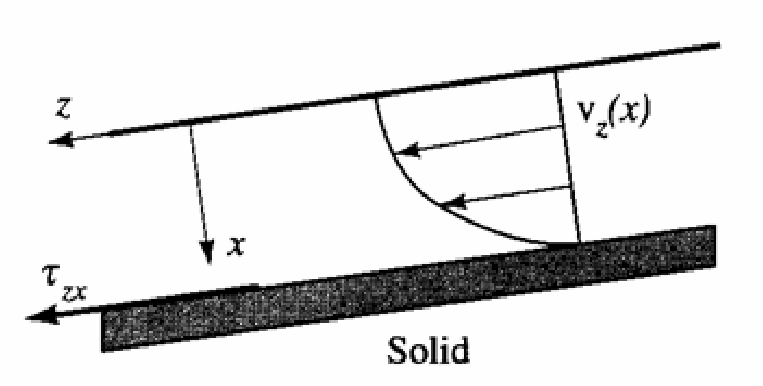
Chapter 3

Forces on, and within, a flowing medium

Shear stress / momentum flux



Force per unit area upon which it acts

$$\tau_{zx} = \frac{\text{force}}{\text{area}} [=] \frac{mL/t^2}{L^2} = \frac{m}{Lt^2}$$

x: direction normal to the surface on which the shear force acts

z: direction at which shear force acts

$$\dot{\gamma} = \frac{\partial v_z}{\partial x} \qquad \tau_{zx} = -\mu\dot{\gamma} = -\mu\frac{\partial v_z}{\partial x}$$

Viscosity: the property of a fluid that determines the ease with which elements of the fluid may be moved relative to one another through the action of some external force.

Newtonian fluid: any fluid that obeys this linear relationship

Momentum flux: the rate at which momentum crosses a boundary per unit area

$$\text{momentum flux} [=] \frac{mL/t}{L^2 t} = \frac{m}{Lt^2}$$

Problem solving and modeling

1. Identify and describe the **phenomenon** of interest.
2. Give a clear statement of the **goals** of the model. (What is expected from the model?)
3. State clearly the **assumptions** you choose to make regarding the physics and the geometry. Your choices define your model.
4. Apply the appropriate **physical** (mechanical, thermodynamical) **principles**.
5. **Solve** the resultant equations.
6. **Compare** the predictions of your model to reality. (Do an experiment, or find a set of appropriate and reliable experimental data.)
7. If necessary, **modify** the assumptions in the hope of improving the degree to which your model mimics reality.

Laminar flow through a tube

The phenomenon

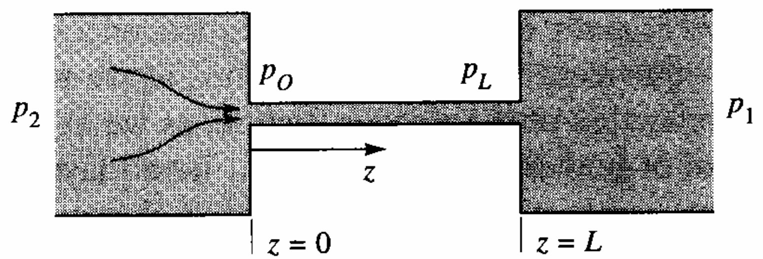


Figure 3.2.1 Liquid flows at a constant rate from one reservoir to the other.

Goal: develop a model for the relationship of the pressure difference to the flowrate.

Assumptions

- The tube is of **uniform circular cross section** along its axis.
- The fluid is **Newtonian** and **incompressible**. (density is not a function of pressure.)
- The flow is **steady state**, **laminar** and **unidirectional** (the velocity vector has only the single component), and **fully developed.** (the flow field does not vary along the tube axis.) $\mathbf{v} = (0, 0, v_z)$ $\quad v_z \neq v_z(z)$ but $v_z = v_z(r)$
- The axis of the tube is **collinear** with the gravity vector.

Physical principles

Conservation of mass and <u>conservation of momentum</u>

(any changes in momentum must be offset by net forces)

Shear force at *r*

$$dF_z\Big|_r = +2\pi r \tau_{zr}\Big|_r dz$$

Shear force at *r+dr*

$$dF_z\Big|_{r+dr} = -2\pi (r+dr)\tau_{zr}\Big|_{r+dr} dz$$

Pressure force at *z*

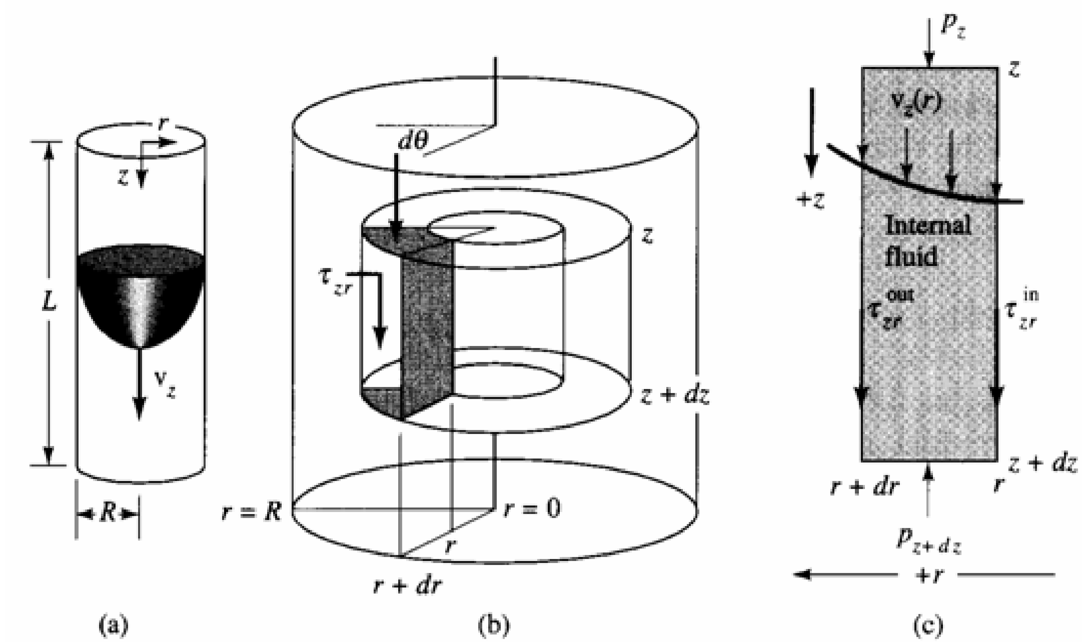
$$+\, p\Big|_z 2\pi r\, dr$$

Pressure force at *z+dz*

$$-\, p\Big|_{z+dz} 2\pi r\, dr$$

Body force

$$\rho g\, 2\pi r\, dr\, dz$$

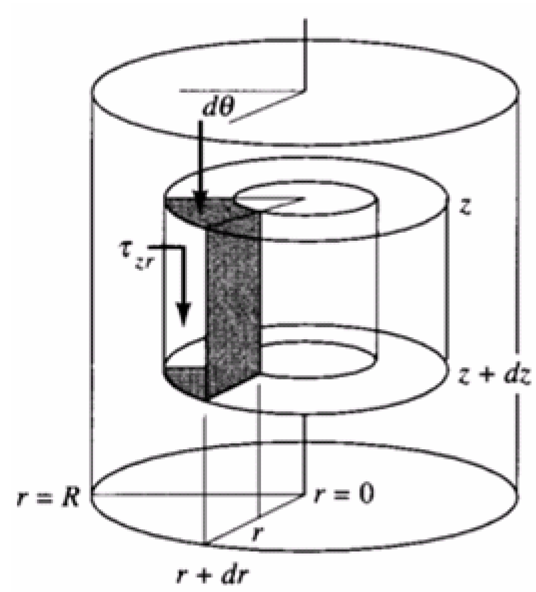


When passing across the boundary of the volume element, from external to internal, if we move in the + direction of the coordinate axis, we use a + sign on the force action on that boundary.

Mass conservation

- differential volume flow rate $dQ = v_z 2\pi r\, dr$
- the rate of flow of mass into the volume $\rho v_z 2\pi r\, dr\big|_z$
- the rate of flow of mass out of the volume $\rho v_z 2\pi r\, dr\big|_{z+dz}$
- the net rate of flow of mass $2\pi r\, dr(\rho v_z\big|_z - \rho v_z\big|_{z+dz}) = 0$

$$\boxed{v_z = \text{constant at a fixed value of } r}$$



Momentum conservation

$$\text{Rate of flow of momentum in}: \rho v_z (\underline{v_z 2\pi r\, dr})\big|_z$$

Differential volume flow rate

$$\text{Rate of flow of momentum out}: -\rho v_z (v_z 2\pi r\, dr)\big|_{z+dz}$$

$$\begin{bmatrix}\text{Sum of forces} \\ \text{in the } z-\text{direction}\end{bmatrix} = \begin{bmatrix}\text{Rate of change of} \\ \text{momentum in the} \\ z-\text{direction}\end{bmatrix}$$

$$dF_z\big|_r + dF_z\big|_{r+dr} + p_z 2\pi r\, dr + (-p\big|_{z+dz} 2\pi r\, dr) + \rho g 2\pi r\, dr\, dz = o$$

$$2\pi\, dz \left[r\tau_{zr}\big|_r - (r+dr)\tau_{zr}\big|_{r+dr} \right] + 2\pi r\, dr \left(p\big|_z - p\big|_{z+dz} + \rho g\, dz \right) = 0$$

$$(r\tau_{zr})_r - (r\tau_{zr})_{r+dr} + r\, dr \left(\frac{(p)_z - (p)_{z+dz}}{dz} + \rho g \right) = 0$$

$$\lim_{dr\to 0} \left\{ \frac{(r\tau_{zr})_r - (r\tau_{zr})_{r+dr}}{r\, dr} \right\} = \lim_{dz\to 0} \left\{ -\left(\frac{(p)_z - (p)_{z+dz}}{dz} + \rho g \right) \right\}$$

Governing equation

$$-\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{zr}) = \frac{\partial p}{\partial z} - \rho g$$

the flow field does not vary down the *z* axis.
velocity gradient and stress are not a function of *z.*

laminar and unidirectional flow means no radial flow.
no pressure variation in the radial direction.
pressure gradient does not depend on *r.*

$$-\frac{1}{r}\frac{d}{dr}(r\tau_{zr}) = C = \frac{dp}{dz} - \rho g$$

$$-(\tau_{zr}) = \frac{Cr}{2} + \frac{E}{r} \qquad p = \rho g z + Cz + G$$

$$\tau_{zr} = -\frac{Cr}{2} \qquad -C = \frac{p_o - p_L}{L} + \rho g$$

$$\boxed{\tau_{zr} = \frac{p_o - p_L + \rho g L}{2L} r}$$

$$\boxed{\tau_{zr} = \frac{p_o - p_L + \rho g L}{2L} r}$$

$$\tau_{zr} = -\mu \frac{d\upsilon_z}{dr}$$

$$\frac{d\upsilon_z}{dr} = -\frac{p_o - p_L + \rho g L}{2\mu L} r = \frac{C}{2\mu} r$$

$$\upsilon_z = \frac{C}{4\mu} r^2 + F$$

No slip boundary condition

$\upsilon_z = 0$ at $r = R$

$$F = -\frac{C}{4\mu} R^2$$

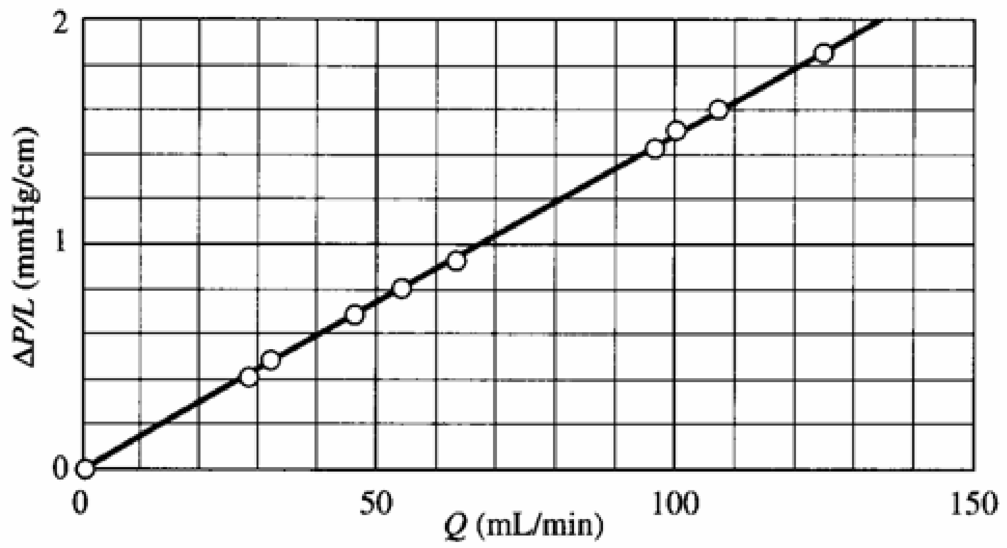
$$\boxed{\upsilon_z = \frac{-CR^2}{4\mu}\left[1 - \left(\frac{r}{R}\right)^2\right]}$$

$$-C = \frac{p_o - p_L}{L} + \rho g$$

$$\boxed{Q = \int_0^R 2\pi \upsilon_z r \, dr = \frac{\pi R^4}{8\mu} \frac{\Delta p}{L}}$$

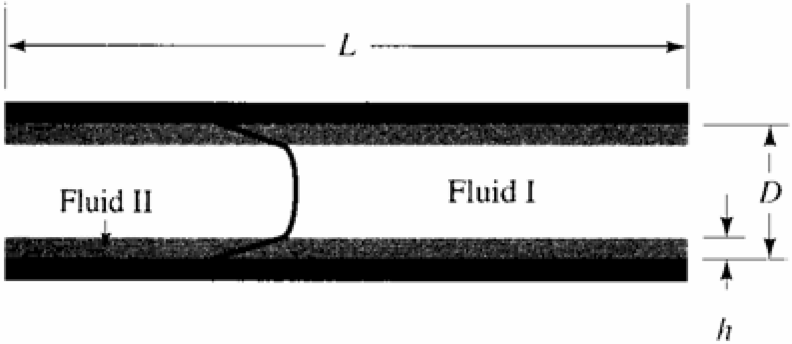
Hagen-Poiseuille equation

$$\Delta p = p_o - p_L + \rho g L$$



Laminar flow through a lubricated tube

to reduce the pressure



$$v_z^{\,I} = \frac{C^I}{4\mu^I} r^2 + F^I \quad over \quad 0 \le r \le \frac{D}{2} - h$$

$$\mu^{II} \frac{dv_z^{\,II}}{dr} = \frac{C^{II}}{2} + \frac{E^{II}}{r} \quad in \quad \frac{D}{2} - h \le r \le \frac{D}{2}$$

$$v_z^{\,II} = \frac{C^{II} r^2}{4\mu^{II}} + \frac{E^{II}}{\mu^{II}} \ln r + F^{II} \quad in \quad \frac{D}{2} - h \le r \le \frac{D}{2}$$

5 integration constants

Single pressure drop is imposed across the ends of the tube

$$C^I = \frac{\Delta P^I}{L} \quad and \quad C^{II} = \frac{\Delta P^{II}}{L}$$

$$C^I = C^{II} = \frac{\Delta P}{L}$$

No slip BC's at wall and at the interface

$$0 = \frac{C^{II} R^2}{4\mu^{II}} + \frac{E^{II}}{\mu^{II}} \ln R + F^{II}$$

$$v_z^{\ II}\Big|_{R-h} = \frac{C^{II}(R-h)^2}{4\mu^{II}} + \frac{E^{II}}{\mu^{II}} \ln(R-h) + F^{II} = v_z^{\ I}\Big|_{R-h} = \frac{C^{I}(R-h)^2}{4\mu^{I}} + F^{I}$$

Continuous shear stress

$$\mu^{II} \frac{dv_z^{\ II}}{dr} = \mu^{I} \frac{dv_z^{\ I}}{dr} \quad at \quad r = R-h \qquad\qquad \frac{C^{I}(R-h)}{2} = \frac{C^{II}(R-h)}{2} + \frac{E^{II}}{(R-h)}$$

Velocity fields

$$\frac{v_z^{\ I}}{\Phi} = \left[(1-\eta)^2 + M\eta(2-\eta)\right] - s^2 \qquad\qquad \frac{v_z^{\ II}}{\Phi} = M(1-s^2)$$

$$\Phi = \frac{-CR^2}{4\mu^I} = \frac{\Delta P R^2}{4\mu^I L}, \quad s = \frac{r}{R}, \quad \eta = \frac{h}{R}, \quad M = \frac{\mu^I}{\mu^{II}}$$

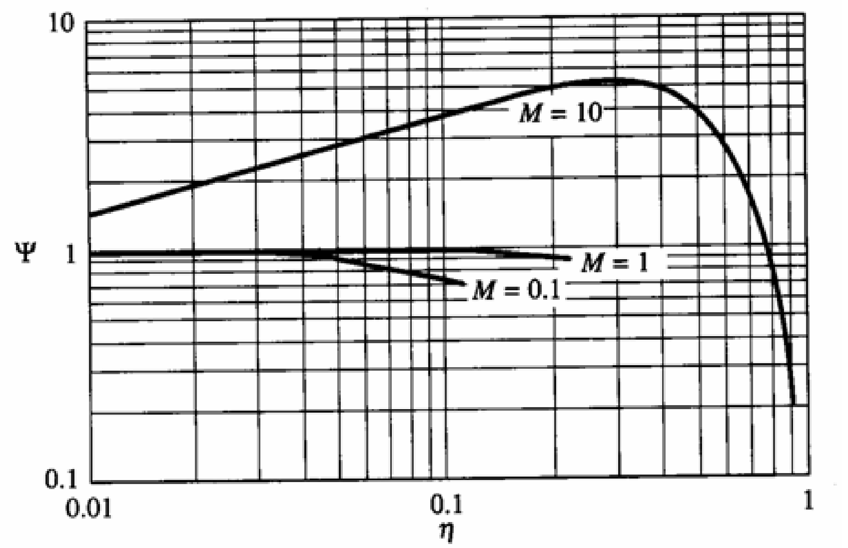
Goal: to investigate the possibility of increasing the flow rate through a tube by providing a lubricating layer of a second fluid at the wall of the tube

$$\frac{Q^I}{2\pi R^2 \Phi} = \frac{(1-\eta)^4}{4} + \frac{M\eta(2-\eta)(1-\eta)^2}{2} \qquad Q^I = \int_0^{R-h} 2\pi r v_z{}^I \, dr$$

The ratio of flow rates with and without the lubricating layer

$$\Psi = \frac{Q^I}{Q^I(\eta = 0)} = (1-\eta)^4 + 2M\eta(2-\eta)(1-\eta)^2$$

$$\eta = \frac{h}{R}, \quad M = \frac{\mu^I}{\mu^{II}}$$



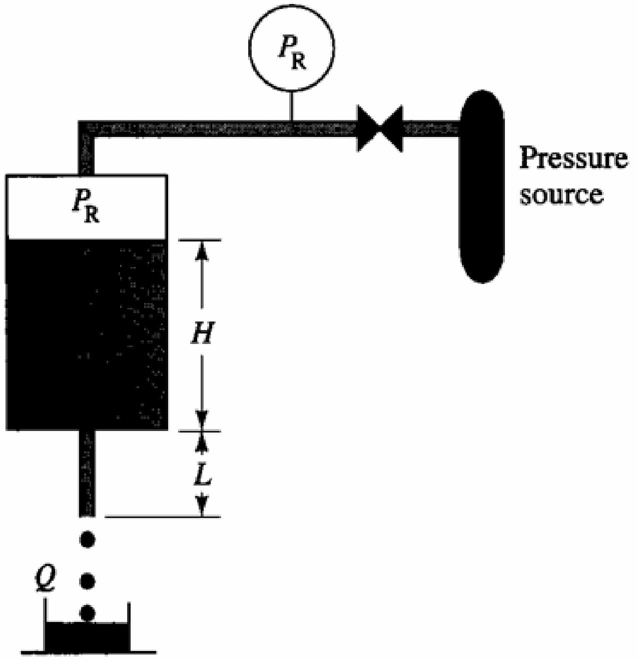
What if the thin film is unstable?

Engineering design

Task: design a capillary viscometer that will be useful for fluids with viscosities of the order of 1000 poise

Interpretation: specify values for *R* and *L* of the capillary, and estimate the required pressure to operate the viscometer

Conceptual design



Design equation

$$\mu = \frac{\pi R^4}{8Q}\frac{\Delta p}{L} \qquad \Delta p = P_{\mathrm{R}} - \rho g H + \rho g L - P_{\mathrm{L}}$$

Constraints: laminar, fully developed, isothermal, Newtonian, <u>no end effect</u>

$$\frac{L_{\mathrm{e}}}{R} = 1.2 + 0.16\frac{Q\rho}{\pi R \mu} = 1.2 + 0.08\,\mathrm{Re}$$

$$\mathrm{Re} = \frac{2Q\rho}{\pi R\mu} = \frac{\rho U D}{\mu}$$

Rough design: order of magnitude estimates

$$\boxed{Q = 1 cm^3/s, \quad R = 0.1 cm}$$

$$\mathrm{Re} = \frac{2(1)(1)}{\pi(0.1)(1000)} = 0.01$$

Fully developed laminar flow

$$\boxed{\frac{L_{\mathrm{e}}}{L} = 0.01} \qquad L = 100 L_e \approx 100 R = 10\, cm$$

Gage pressure: relative to atmospheric pressure

$$\Delta p = \frac{8\mu QL}{\pi R^4} = \frac{8(1000)(1)(10)}{\pi(0.1)^4}$$

$$\Delta p = \frac{8}{\pi}\times 10^8\ \text{dyn/cm}^2 = \frac{800}{\pi}\ \text{atm}$$

$$= 4000\ \text{psi} \quad (\text{all numbers rough, because we want quick estimates})$$

Evaluation: based on these rough numbers, do we need to modify any of the choices we made? **Pressure is too high**

$$\Delta p = \frac{8\mu Q}{\pi}\frac{L}{R}\frac{1}{R^3}$$

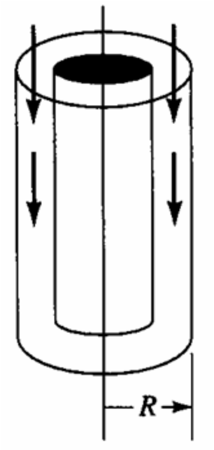
As pressure is sensitive to *R*, increase *R* by a factor of 2 keeping *L*/*R*=100

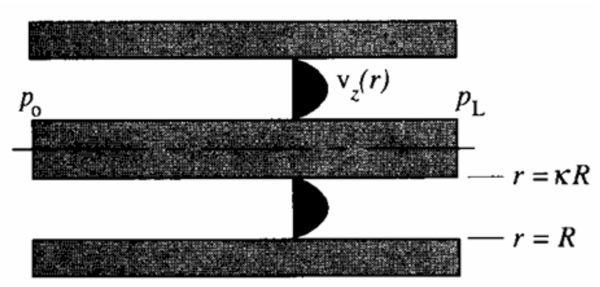
$$\Delta P = \frac{4000}{8} = 500\ psi$$

$$\rho g H \approx \rho g L = (1)(10^3)(20) = 2\times 10^4\ \text{dyn/cm}^2 \approx 0.3\ \text{psi}$$

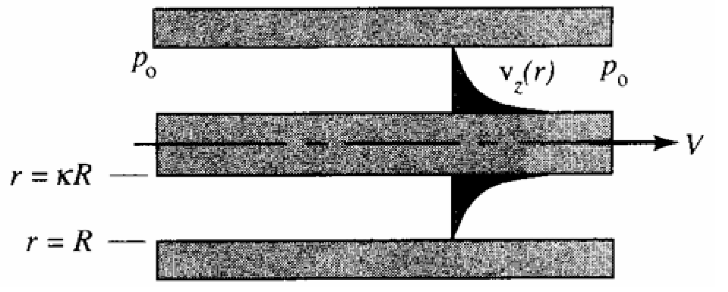
Hydrostatic effects are negligible

Annular flow

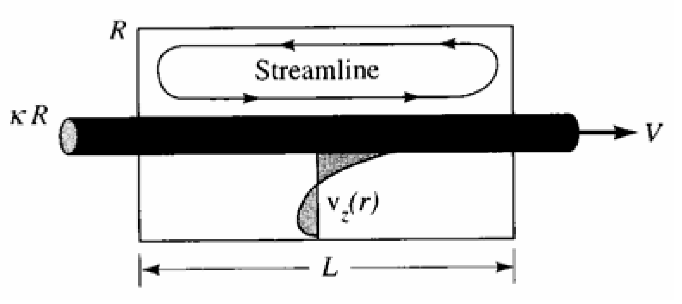




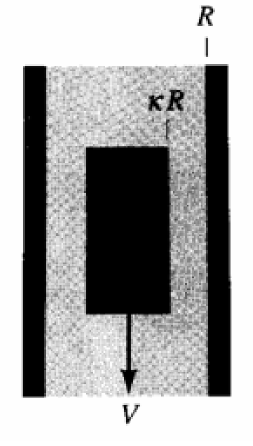
Pressure driven flow



Drag flow

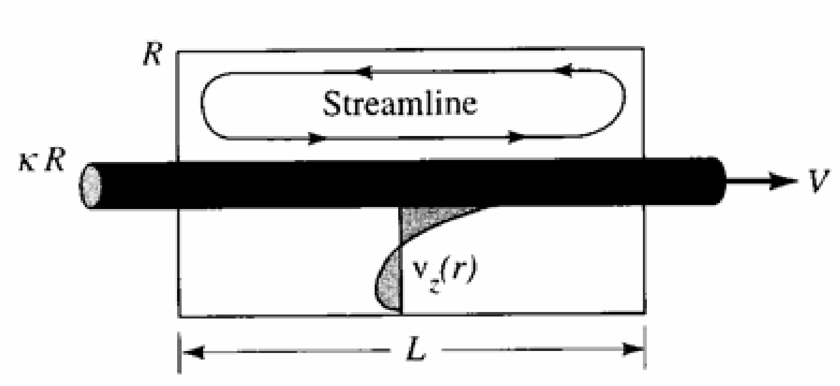


Confined flow between two cylinders



Motion of a cylinder of finite length

Annular flow in a closed container



Ignore the flow at the ends
Over most of the length, the flow is strictly axial laminar flow (*L*>>*R*)

$$-\frac{1}{r}\frac{d}{dr}(r\tau_{zr}) = \frac{\Delta P}{L} \qquad \tau_{zr} = -\mu\frac{d\upsilon_z}{dr} \qquad \begin{array}{l} \upsilon_z = V \quad \text{at} \quad r = \kappa R \\ \upsilon_z = 0 \quad \text{at} \quad r = R \end{array}$$

$$\upsilon_z(r) = \frac{-\Delta P R^2}{4\mu L}\left(1-\frac{r^2}{R^2}\right) + \frac{\ln(R/r)}{\ln(1/\kappa)}\left[V + \frac{\Delta P R^2}{4\mu L}(1-\kappa^2)\right]$$

$$\frac{\upsilon_z(r)}{V} = \varphi(s) \qquad \frac{r}{R} = s \qquad \frac{\Delta P R^2}{4\mu L V} = \Phi$$

$$\varphi(s) = -\Phi(1-s^2) + \frac{\ln s}{\ln\kappa}\left[1+\Phi(1-\kappa^2)\right]$$

No net flow across any surface normal to the z axis

$$Q = 2\pi \int_{\kappa R}^{R} \upsilon_z(r)\, r\, dr = 0 \qquad \int_{\kappa}^{1} \varphi(s)\, s\, ds = 0$$

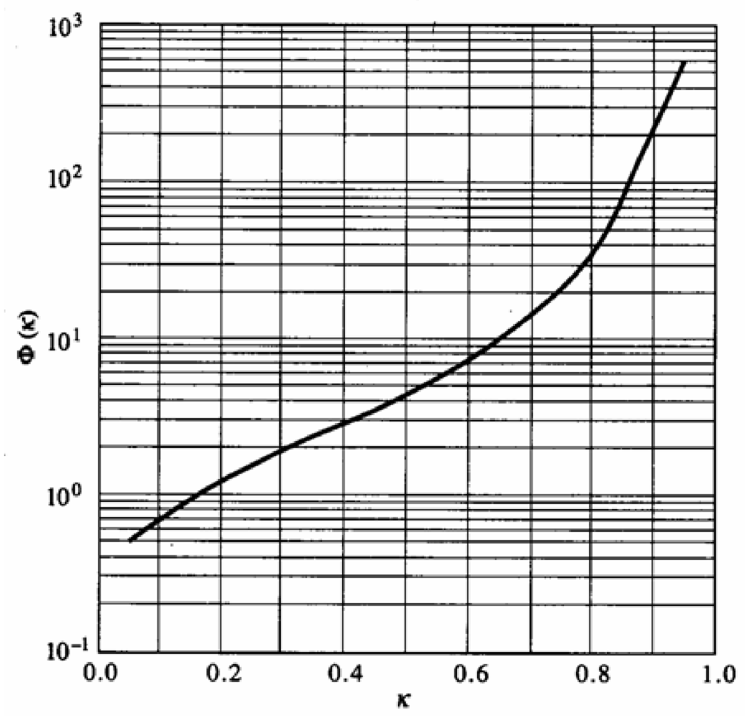
$$0 = -\Phi \int_{\kappa}^{1} (1 - s^2)\, s\, ds + \left[1 + \Phi(1 - \kappa^2)\right] \frac{1}{\ln \kappa} \int_{\kappa}^{1} (\ln s)\, s\, ds$$

$$0 = F(\Phi, \kappa) \quad \text{or} \quad \Phi = G(\kappa)$$

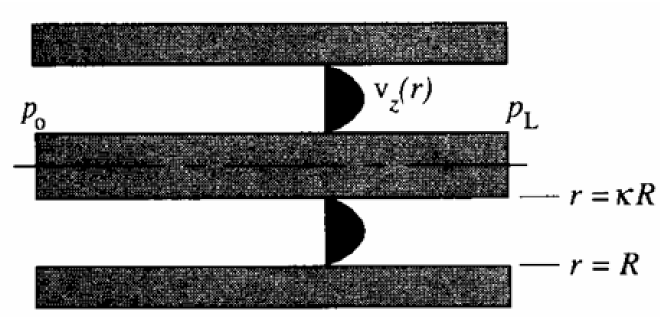
$$\Phi(\kappa) = -\frac{1 - \kappa^2 + 2\kappa^2 \ln \kappa}{(1 - \kappa^4) \ln \kappa + (1 - \kappa^2)^2}$$

$$\kappa \to \Phi(\kappa) \to \Delta P$$

As a function of operating parameter (*V*) and design parameters (*R*,*L*)



Annular flow in an open tube



$$-\frac{1}{r}\frac{d}{dr}(r\tau_{zr}) = \frac{\Delta P}{L} \qquad \tau_{zr} = -\mu\frac{d\upsilon_z}{dr}$$

$$\upsilon_z = 0 \quad \text{at} \quad r = \kappa R$$

$$\upsilon_z = 0 \quad \text{at} \quad r = R$$

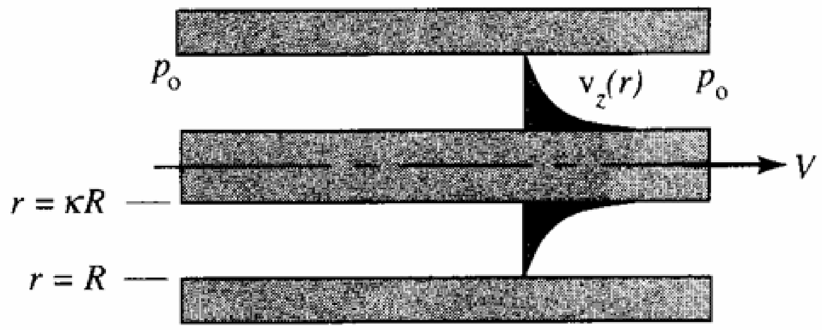
$$\upsilon_z(r) = \frac{\Delta P R^2}{4\mu L}\left[(1-s^2) - (1-\kappa^2)\frac{\ln s}{\ln \kappa}\right] \qquad s = r/R$$

Nondimensional velocity function

$$\varphi_{\mathrm{p}}(s) = \frac{4\mu L \upsilon_z(r)}{\Delta P R^2} \qquad \varphi_{\mathrm{p}}(s) = (1-s^2) - (1-\kappa^2)\frac{\ln s}{\ln \kappa}$$

$$Q_{\mathrm{p}} = \frac{\pi \Delta P R^4}{8\mu L}\left[1-\kappa^4 + \frac{(1-\kappa^2)^2}{\ln \kappa}\right] \qquad Q = 2\pi\int_{\kappa R}^{R} \upsilon_z(r)\, r\, dr$$

Annular drag flow



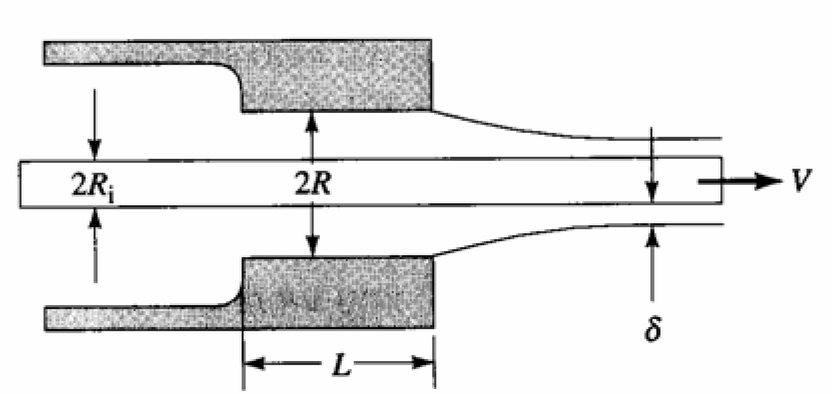
Annular flow in a closed container

$$\upsilon_z(r) = \frac{-\Delta P R^2}{4\mu L}\left(1 - \frac{r^2}{R^2}\right) + \frac{\ln(R/r)}{\ln(1/\kappa)}\left[V + \frac{\Delta P R^2}{4\mu L}(1-\kappa^2)\right]$$

$$\Delta P = 0$$

$$\upsilon_z(r) = V\frac{\ln s}{\ln \kappa} \qquad Q_d = \pi R^2(1-\kappa^2)V\left[\frac{2\kappa^2\ln\kappa + 1 - \kappa^2}{-2(1-\kappa^2)\ln\kappa}\right]$$

Design of a wire coating die



Axial annular drag flow

Steady state, isothermal, Newtonian

Goal: to find the relationship between the downstream coating thickness and the other parameters that characterize the performance of the system

Mass flow rate of coating $\qquad \dot{m} = \rho\pi[(\kappa R+\delta)^2 - \kappa^2 R^2]V$

Mass flow rate due to the drag flow through the die $\qquad \dot{m} = \rho' Q$

Density of the fluid within the die

$$R - R_i << L \qquad Q_d = \pi R^2(1-\kappa^2)V\left[\frac{2\kappa^2 \ln\kappa + 1 - \kappa^2}{-2(1-\kappa^2)\ln\kappa}\right]$$

$$\delta'^2 + 2\delta' - \frac{\rho'}{\rho}\frac{1-\kappa^2}{\kappa^2}H(\kappa) = 0$$

$$\delta' = \frac{\delta}{R_\mathrm{i}} = \left[1 + \frac{\rho'}{\rho}\frac{1-\kappa^2}{\kappa^2}H(\kappa)\right]^{1/2} - 1$$

Independent of wire velocity

Thickness can be varied only through changes in the geometry of the die.

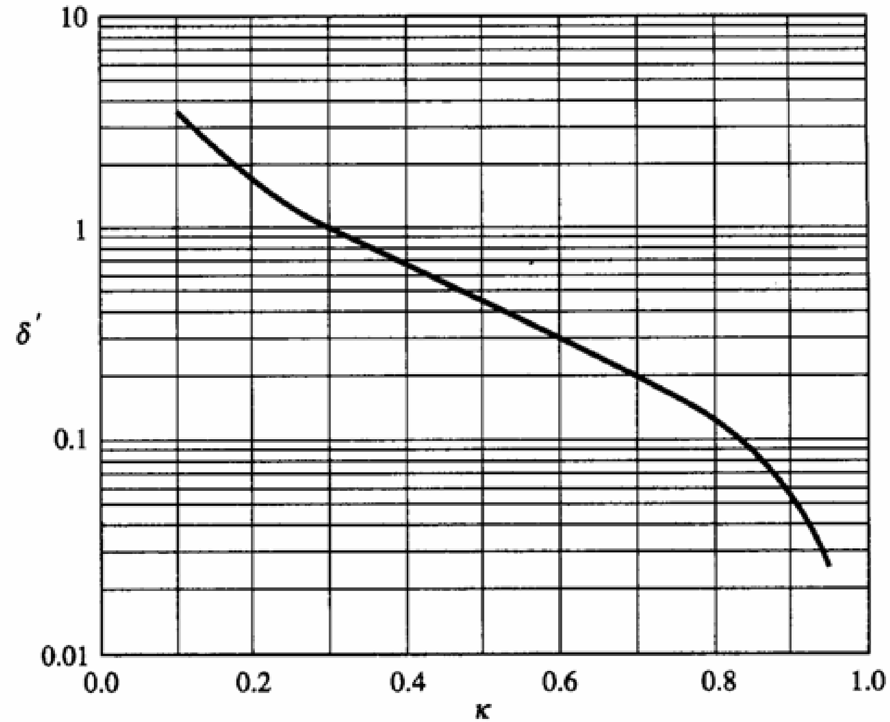
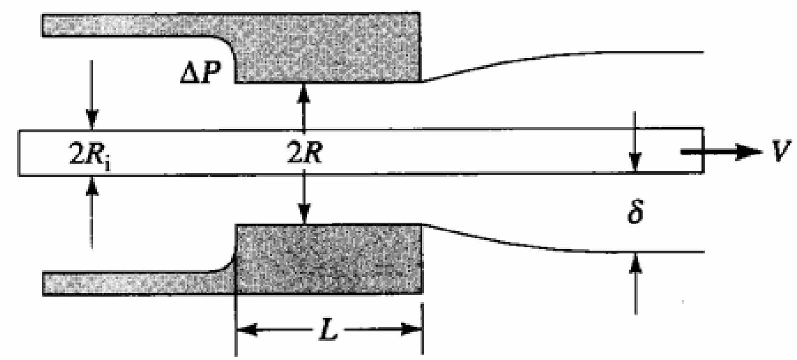
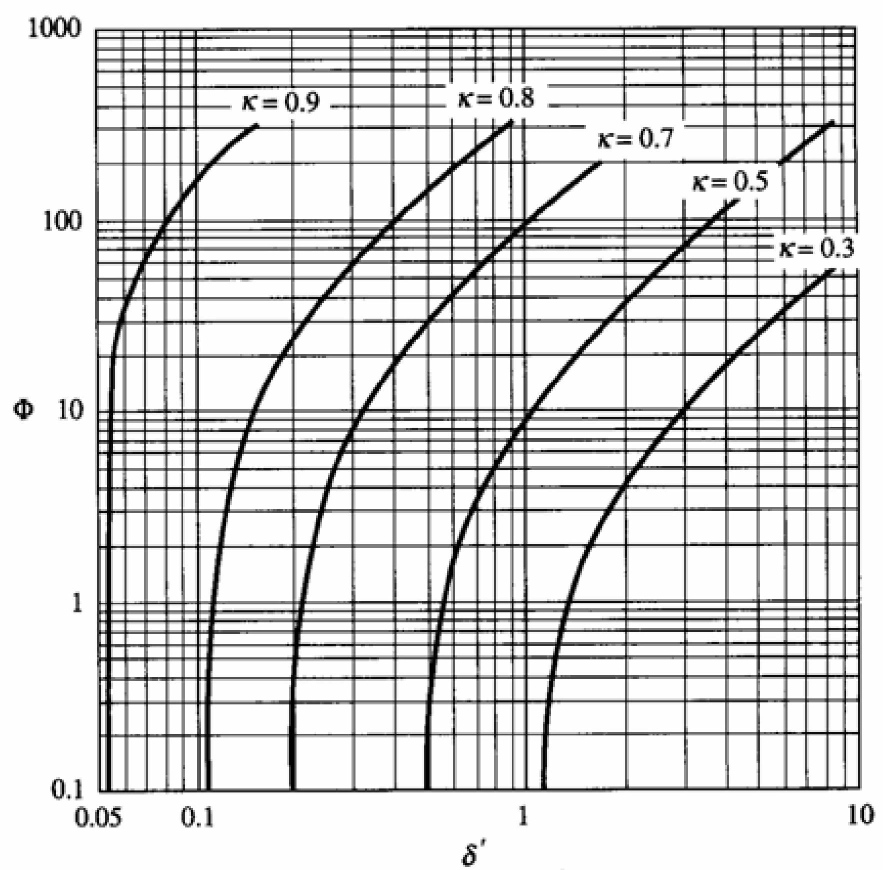


Figure 3.3.10 Dimensionless coating thickness in wire coating (no pressure) for $\rho/\rho' = 1$.

What if coating thickness is not exactly at the desired level?

- impose a postive pressure on the fluid upstream of the die





Pressure driven + Drag

Interpretation of the graph

$$Q = Q_d + Q_p = \frac{\pi R^2 V}{2}\left(\frac{1-\kappa^2}{\ln(1/\kappa)} - 2\kappa^2\right) + \frac{\pi \Delta P R^4}{8\mu L}\left[1-\kappa^4 - \frac{(1-\kappa^2)^2}{\ln(1/\kappa)}\right]$$

Maximum shear stress in a wire coating die

$$\upsilon_z(r) = \frac{\Delta P R^2}{4\mu L}\left[(1-s^2) - (1-\kappa^2)\frac{\ln s}{\ln \kappa}\right] + V\frac{\ln s}{\ln \kappa}$$

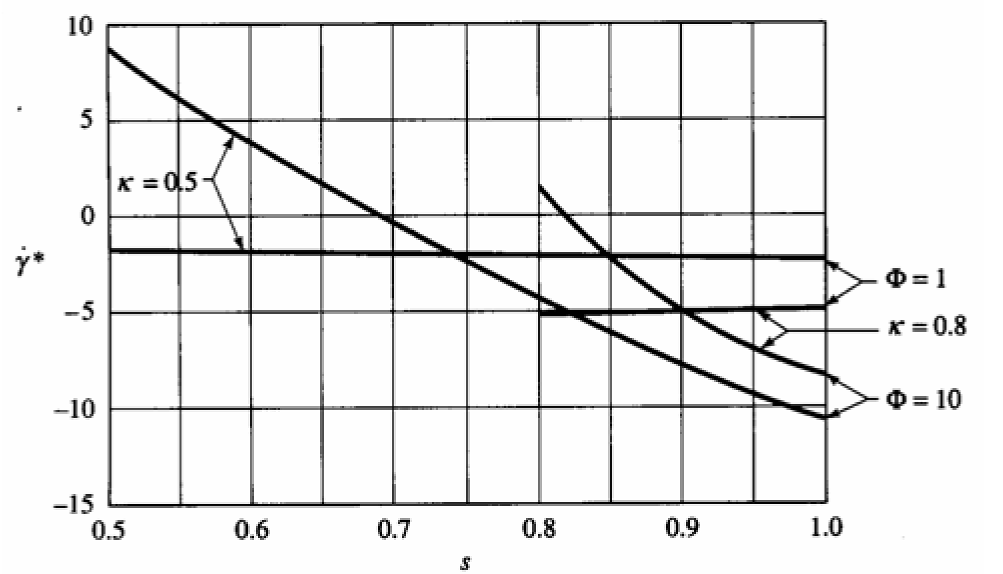
$$\dot{\gamma} = \frac{\partial \upsilon_z(r)}{\partial r} = \frac{\Delta P R}{4\mu L}\left[-2s - \frac{1-\kappa^2}{s\ln\kappa}\right] + \frac{1}{R}\frac{1}{s\ln\kappa}$$

$$\dot{\gamma}^* = \frac{R}{V}\dot{\gamma} = -\Phi\left[2s + \frac{1-\kappa^2}{s\ln\kappa}\right] + \frac{1}{s\ln\kappa}$$

for small pressure, drag flow

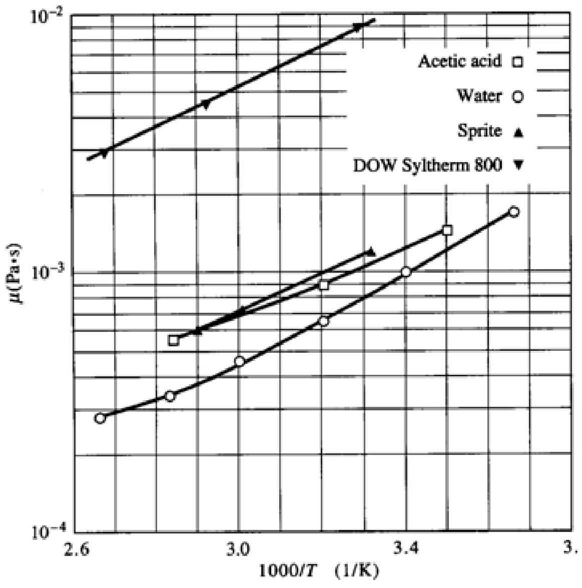
for large pressure, shear rate changes sign at the point of maximum velocity

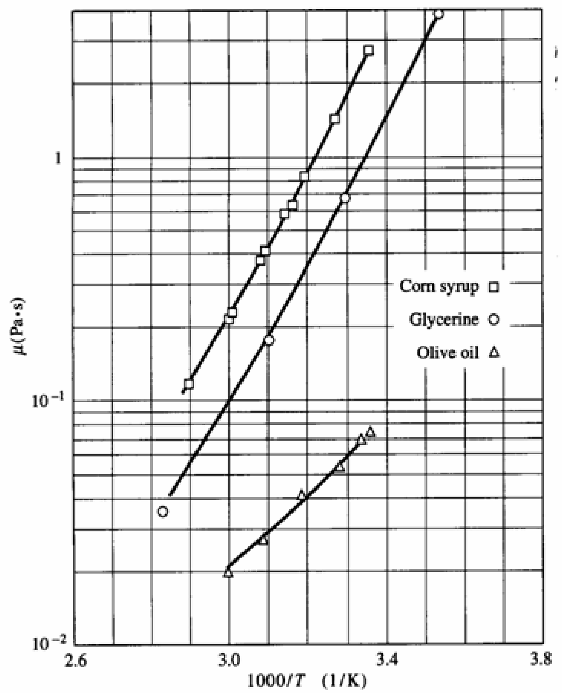
$$\tau = \mu\dot{\gamma}$$

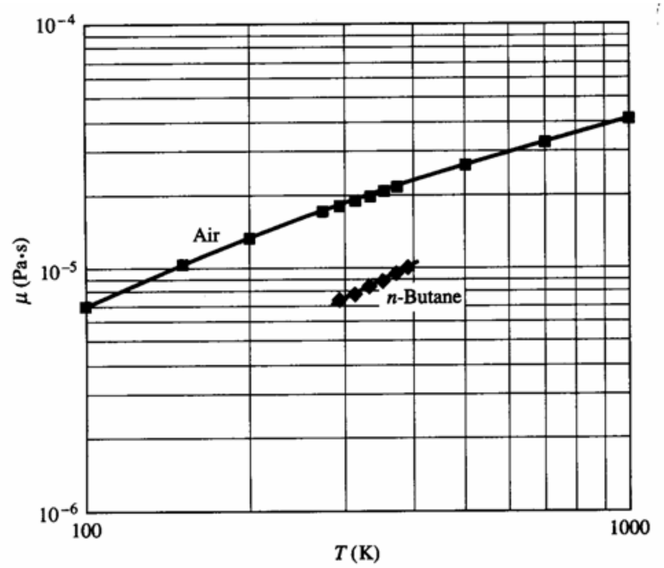


The viscosity of fluids

The resistance a fluid exhibits to being deformed by the imposition of stresses
unit: centipoise, Pa.s



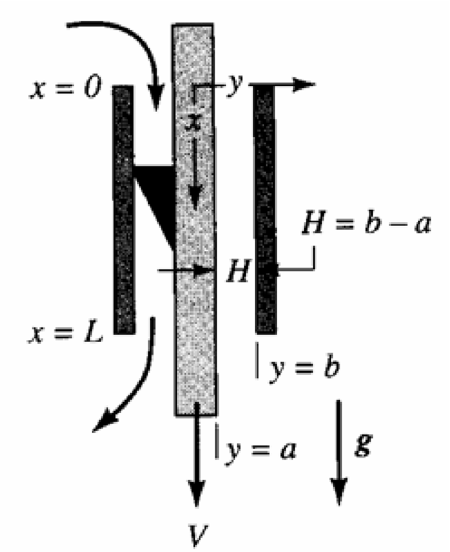




$$\mu = A e^{B/T}$$

Motion of a planar sheet through a submerged restriction

Completely immersed in a large body of viscous liquid



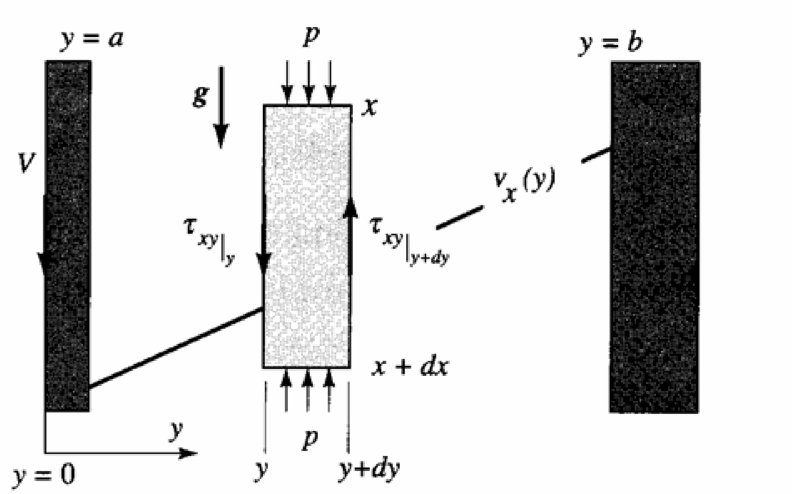


Figure 3.5.2 Control volume for momentum balance.

Neglect the flow near the entrance and exit
Unidirectional laminar, fully developed, Newtonian

$$\mathbf{v}(y) = (v_x(y), 0, 0) \qquad \tau_{xy}(y) = -\mu \frac{dv_x(y)}{dy}$$

Force balance

$$p_x \, dy\, dz - p_{x+dx} \, dy\, dz + \tau_{xy_x} \, dx\, dz - \tau_{xy_{x+dx}} \, dx\, dz + \rho g \, dx\, dy\, dz = 0$$

Divide by the volume dxdydz and take the limit as the volume shrinks to a point within the fluid

$$-\frac{\partial p}{\partial x} + \rho g = \frac{\partial \tau_{xy}}{\partial y}$$

$$-\frac{dp}{dx} + \rho g = C$$

$$p(x) = \rho g x - Cx + D$$

No information on entrance pressure, and just assume

$$p_0 = P_{\text{stat}} \quad \text{at} \quad x = 0$$

$$p_{\text{L}} = P_{\text{stat}} + \rho g L \quad \text{at} \quad x = L$$

$$\boxed{p(x) = \rho g x + P_{stat}}$$

$$\frac{d\tau_{xy}}{dy} = C$$

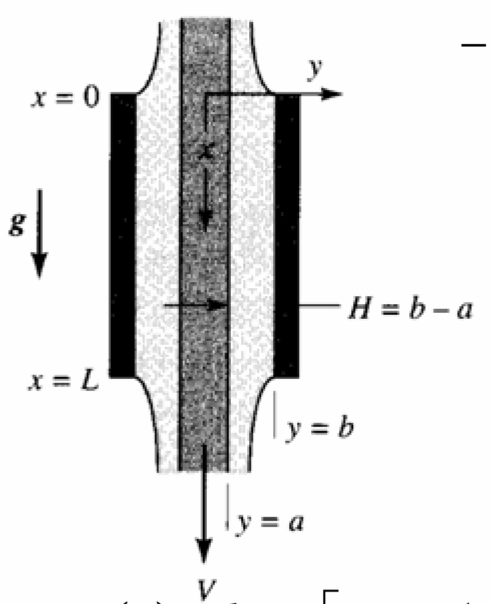
$$\tau_{xy} = \text{constant} = d$$

$$\boxed{\frac{\upsilon_x(y)}{V} = \frac{b-y}{b-a}}$$

Gravity affects no role in altering the nature of the flow field, but simply affects the pressure distribution

Motion of a wetted planar sheet through a restriction

Moving sheet is coated with a thin flim of viscous liquid



$$-\frac{\partial p}{\partial x} + \rho g = \frac{\partial \tau_{xy}}{\partial y} \qquad -\frac{dp}{dx} + \rho g = C \qquad \frac{d\tau_{xy}}{dy} = C$$

$$p(x) = \rho g x - Cx + D$$

$$p = P_{\text{atm}} \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L$$

Pressure boundary condition

$$C = \rho g \qquad \tau_{xy} = \mathrm{C}y + E = \rho g y + E$$

$$\boxed{\frac{\upsilon_x(y)}{V} = \frac{b - y}{b - a} + \frac{\rho g}{2\mu V}[b^2 - y^2 - (a + b)(b - y)]}$$

$$\frac{\upsilon_x(y)}{V} = \frac{b - y}{b - a}\left[1 + \frac{\rho g (a - b)^2}{2\mu V}\left(\frac{y - a}{b - a}\right)\right]$$

$$\beta = \frac{\rho g (b - a)^2}{2\mu V} = \frac{\rho g (b - a) WL}{[2\mu V/(b - a)]WL}$$

If the weight of the confined liquid is very small compared to the shear force, the velocity profile is close to that of the fully submerged case, and gravity does not alter the flow field.