Chapter 2

Statics, dynamics, and surface tension

Hydrostatics

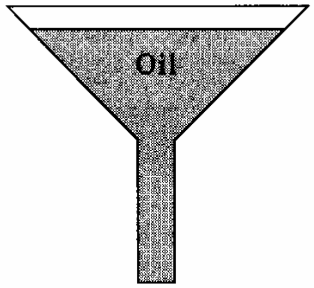
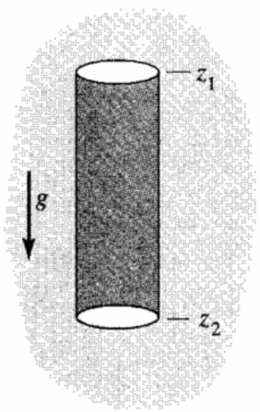


Figure 2.1.1 Oil in a closed funnel.



$$F_{up} = p_2 A = p_1 A + \rho A h g = F_{down}$$

$$p_2 - p_1 = \rho g h$$

Fundamental law of hydrostatics for an incompressible fluid
Pressure increases linearly with position in the direction of gravity.

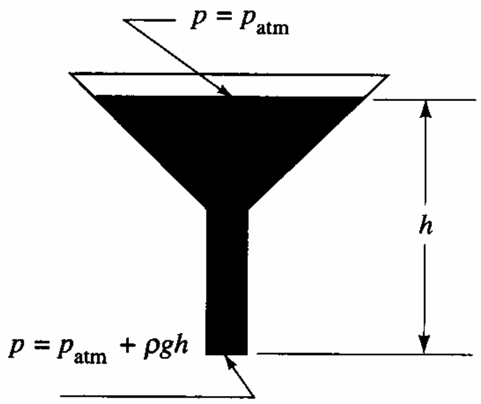


Figure 2.1.5 Closed funnel, no flow

What happens
if we unplug the bottom of the funnel

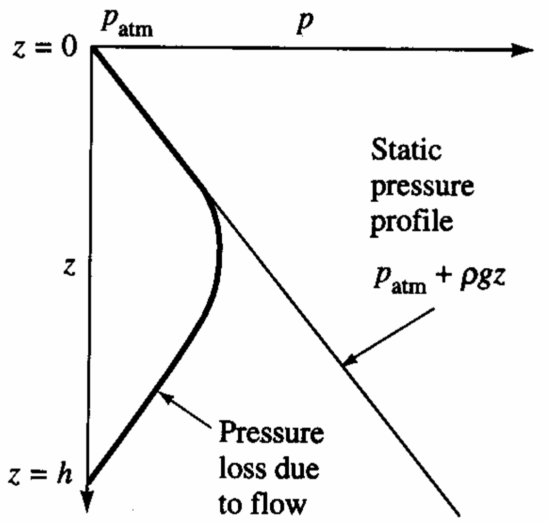
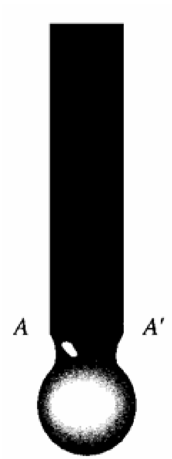


Figure 2.1.6 Pressure distribution with and without flow (static pressure profile).

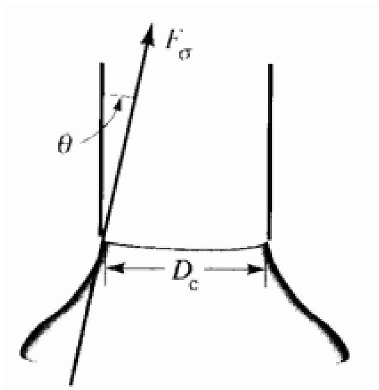
Suppose that the exit flow is very slow because the liquid is very viscous or because the exit tube has a very small diameter, and that a drop begins to form slowly at the end of the capillary.

Then it is possible that the flow will stop.

Surface tension



As the drop grows, its weight will overcome the force of surface tension that holds the drop to the tip of the capillary, and the drop will fall off.



Along the three phase contact line, surface tension is the force per unit length acting tangentially to the liquid-gas interface.

There is a vertical component of force

$$F_\sigma \cos\theta = \sigma \pi D_c \cos\theta$$

If the drop is static, it balances the weight of the drop

$$\sigma \pi D_c \cos\theta = \rho g V$$

First modeling

Determine the maximum size of a drop that will hang on a capillary

$$\sigma \pi D_c \cos\theta = \rho g V$$

If the drop volume changes slightly, what happens on the lhs?

As the drop gets heavier, the angle approaches zero, and lhs reaches its maximum. Then the drop will fall when its volume exceeds

$$V_{max} = \frac{\sigma \pi D_c}{\rho g}$$

If the drop is nearly a sphere

$$V = \frac{\pi D_d^3}{6} \qquad D_d = \left(\frac{6\sigma D_c}{\rho g}\right)^{1/3} \qquad \frac{D_d}{D_c} = \left(\frac{6\sigma}{D_c^2 \rho g}\right)^{1/3}$$

$$\boxed{\frac{D_d}{D_c} = \left(\frac{6}{\mathrm{Bo}}\right)^{1/3} = 1.82\,\mathrm{Bo}^{-1/3}}$$

$$\mathrm{Bo} = \frac{D_c^2 \rho g}{\sigma}$$

Bond number: determines the nature of the interaction of gravitational and surface forces

Test of the model

$$\frac{D_d}{D_c} = \left(\frac{6}{Bo}\right)^{1/3} = 1.82\,Bo^{-1/3}$$

$$N_d = \left(\frac{D_d}{D_c}\right)Bo^{1/3}$$

$$We = \frac{\rho(4Q/\pi D_{ci}^2)^2 D_{co}}{\sigma}$$

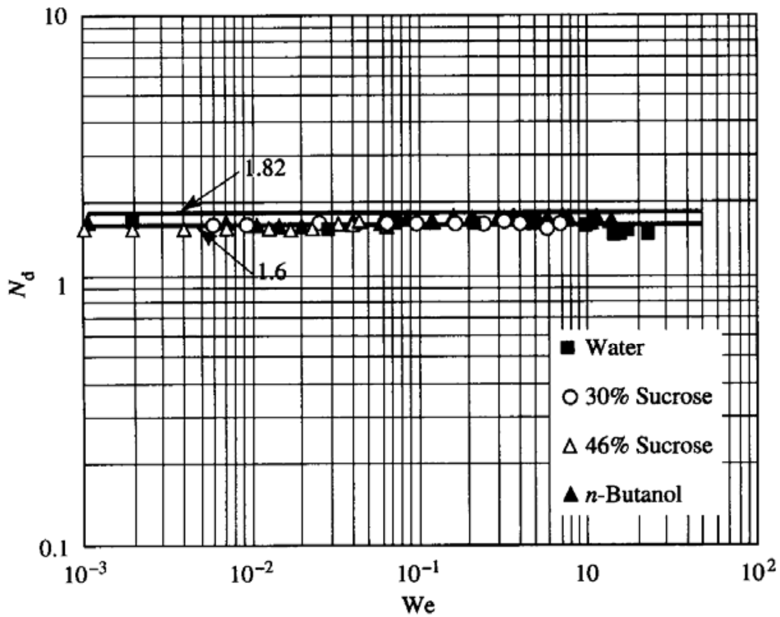
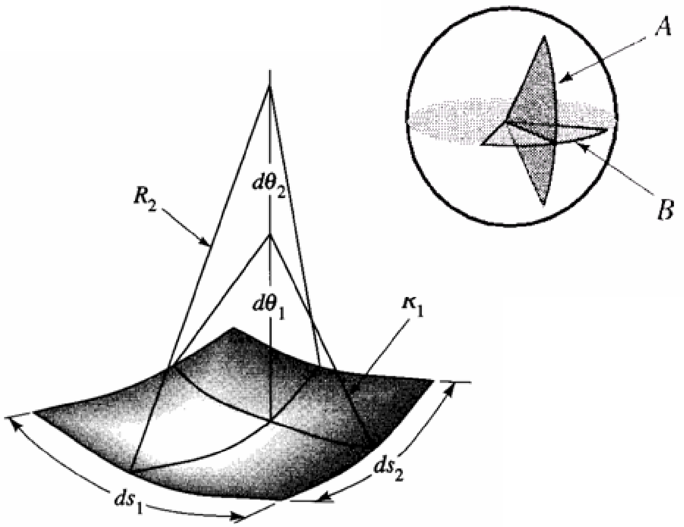


Figure 2.1.10 Dimensionless drop size (see Eq. 2.1.12) for several liquids, as a function of the Weber number.

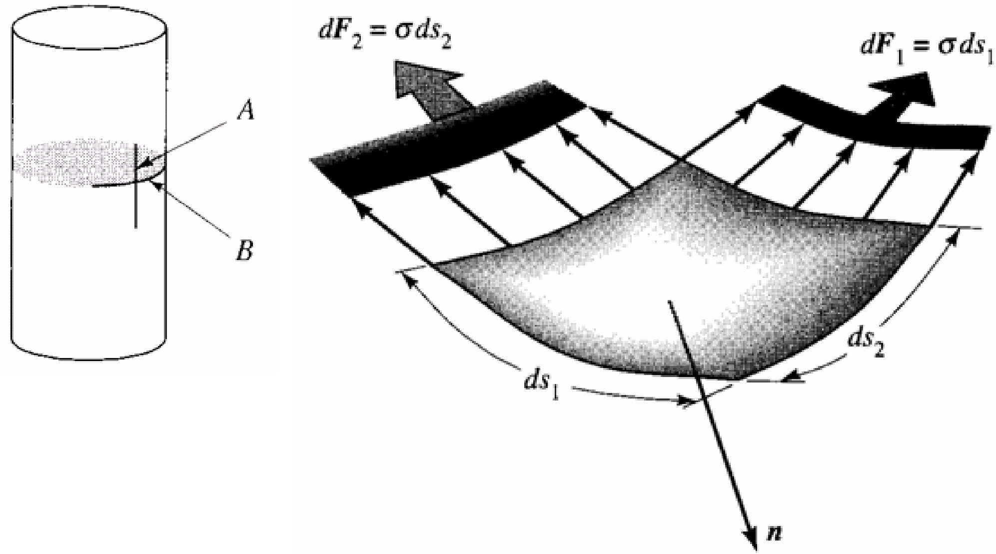
The model captures the essential physics of the process, failing only to yield the exact value of the coefficient.
Because it leaves residual liquid attached to the tip when the drop falls from the capillary tip.

Capillary hydrostatics

1. Interfacial tension can give rise to a pressure difference across the interface.
2. The pressure difference can be calculated from the shape and size of the interface.

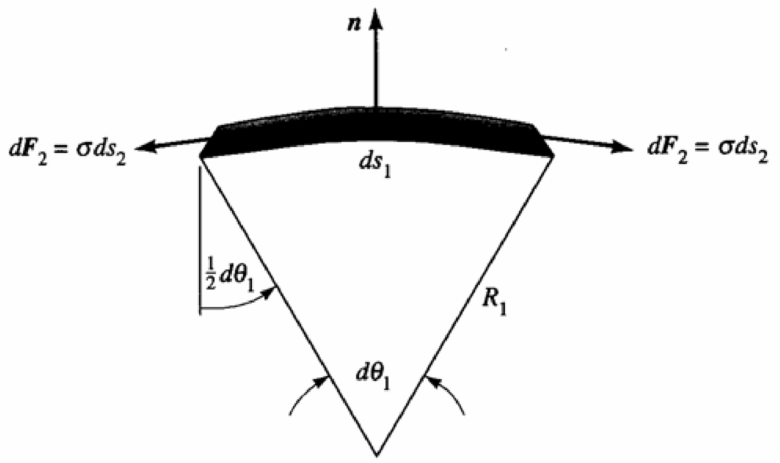


For any surface, there is a pair of orthogonal curves at any point on the surface.



The interfacial tension is a force per unit length acting across a line element in a surface, the line of action of the force being normal to the line and tangential to the surface.

Young-Laplace equation



Along each of the four arcs that define the differential area, a distributed force acts uniformly.

Sum the force components of these four forces in the direction normal to the surface.

n component of the force along ds_2

$$dF_2 = 2(\sigma\, ds_2)\sin(d\theta_1/2) = \frac{\sigma}{R_1} ds_1 ds_2$$

$$ds_1 = R_1 d\theta_1 \qquad \sin\left(\tfrac{1}{2} d\theta_1\right) = \tfrac{1}{2} d\theta_1$$

for small angle

$$dF_1 = \frac{\sigma}{R_2} ds_1 ds_2 \qquad dF_\sigma = \sigma\, ds_1 ds_2 \left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Static force balance

$$\left[p_o - p_i + \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right] ds_1 ds_2 = 0$$

$$p_i - p_o = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Pressure inside a gas bubble

For a spherical gas bubble $R_1 = R_2 = R$ $\Delta p = \dfrac{2\sigma}{R}$

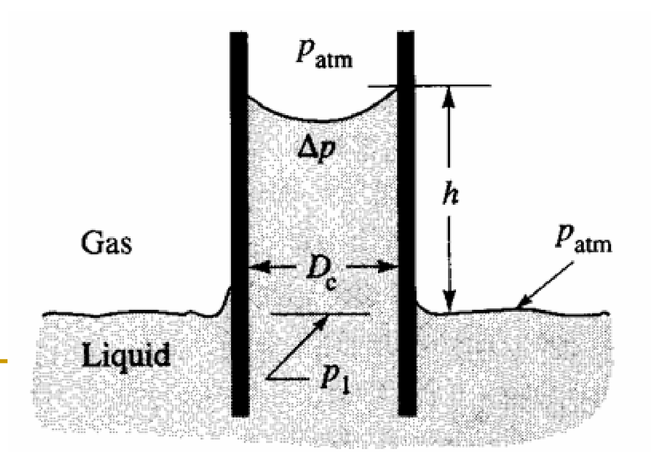
Typical surface tension of a liquid is 0.05 N/m

For a bubble of radius 1mm $$\Delta p = \frac{2(0.05)}{0.001} = 100\ \mathrm{N/m^2}$$

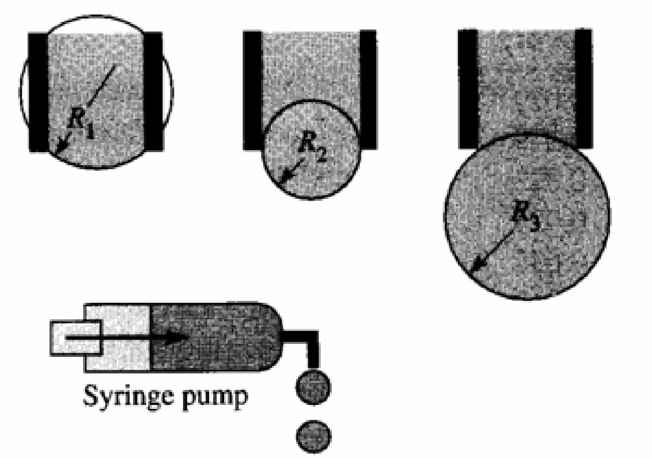
For a bubble of radius 1μm $$\Delta p = \frac{2(0.05)}{1\times 10^{-6}} = 1\times 10^{5}\ \mathrm{N/m^2} \approx 1 atm$$

<u>Exercises</u>

Liquid rise in a capillary



Back-pressure in a growing drop



Pressure in response to external forces

Figure 2.3.1 A container of liquid rotates about its axis.

Differential force balance in z direction

$$dp_z = -\rho g\, dz$$

$$p_2 - p_1 = \rho g h$$

Differential force balance in *r* direction

$$dp_r\, r d\theta dh = dm \times a = dm\, r\omega^2 = \rho\, r d\theta dr dh\, r\omega^2$$

$$dp_r = \rho r \omega^2\, dr$$

$$dp = \frac{\partial p}{\partial z} dz + \frac{\partial p}{\partial r} dr = dp_z + dp_r$$

Pressure field within a rotating body of liquid

$$p = -\rho g z + \frac{\rho r^2 \omega^2}{2}$$

Surface of a rotating liquid

Set atmospheric pressure to zero $p = -\rho g z + \frac{\rho r^2 \omega^2}{2}$

$$Z(r) = \frac{r^2 \omega^2}{2g}$$

Pressure in an accelerating liquid: shape of the free surface

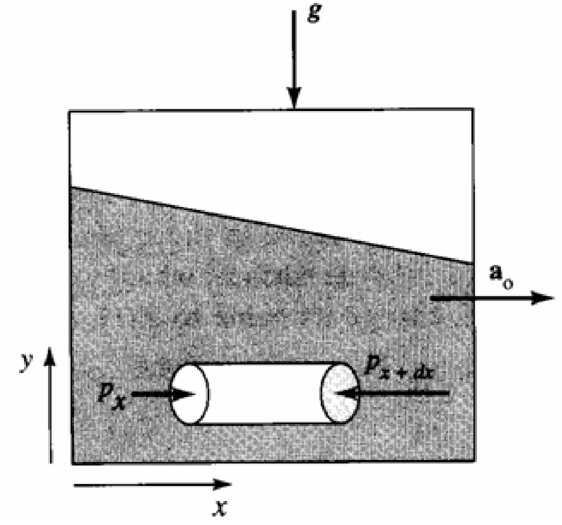
Liquid container moves in the +*x* direction at a velocity that increases linearly with time

$$p_x dA - p_{x+dx} dA = \rho (dA)(dx) a_o$$

$$\frac{\partial p}{\partial x} = -\rho a_o \qquad p(x) = -\rho a_o x + A(y)$$

$$\frac{\partial p}{\partial y} = -\rho g \qquad p(y) = -\rho g y + B(x)$$

$$p(x,y) = -\rho a_o x - \rho g y + D$$



Atmospheric pressure at surface *Y(x)*

$$Y(x) = -\frac{(p_o - D) + \rho a_o x}{\rho g}$$