Chap. 10. Fourier Series, Integrals, and Transforms

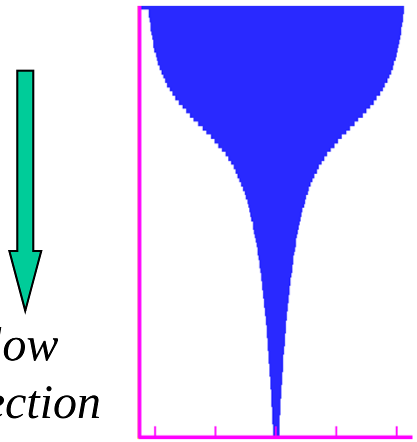
- Fourier series: series of cosine and sine terms
  - for general periodic functions (even discontinuous periodic func.)
  - for ODE, PDE problems

  (more universal than Taylor series)

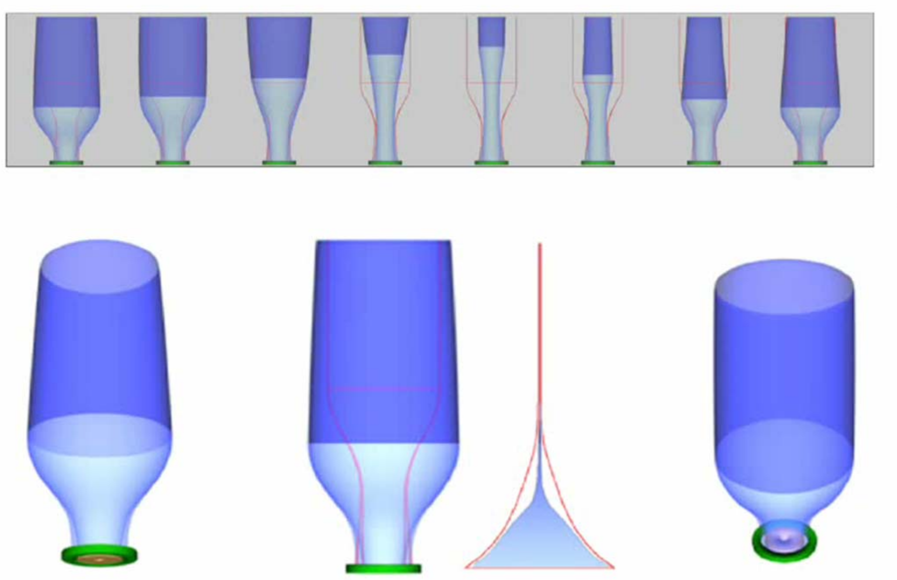
10.1. Periodic Functions. Trigonometric Series

- Periodic function: $f(x+p) = f(x)$ for all x; period=p
- Examples) *Periodic instabilities in rheological processes*

Spinning process



Tubular film blowing process



- f(x + np) = f(x) for all x (n: integer)

- h(x) = af(x) + bg(x) (f & g with period p; a & b: constants) → h(x) with period p.

Trigonometric Series

- Trigonometric series of function f(x) with period p=2π:

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$ (a_k,b_k: constant coefficients)

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

These series converge,
its sum will be a function of period 2π
→ ***Fourier Series***

10.2. Fourier Series

- Representation of periodic function f(x) in terms of cosine and sine functions

Euler Formulas for the Fourier Coefficients

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$ *(Periodic function of period 2π)*

(1) Determination of the coefficient term a_0

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]dx$$

$$= \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi}\cos nx\, dx + b_n \int_{-\pi}^{\pi}\sin nx\, dx\right) = 2\pi a_0$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

(2) Determination of the coefficients a_n of the cosine terms

$$\int_{-\pi}^{\pi} f(x)\cos mx\, dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx\, dx$$

$$= \int_{-\pi}^{\pi} a_0 \cos mx\, dx + \left[\sum_{n=1}^{\infty} a_n\left(\frac{1}{2}\int_{-\pi}^{\pi}\cos(n+m)x\, dx + \frac{1}{2}\int_{-\pi}^{\pi}\cos(n-m)x\, dx\right)\right]$$

(Except n=m)

$$+\left[\sum_{n=1}^{\infty} a_n\left(\frac{1}{2}\int_{-\pi}^{\pi}\sin(n+m)x\, dx + \frac{1}{2}\int_{-\pi}^{\pi}\sin(n-m)x\, dx\right)\right]$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\, dx \quad (m = 1, 2, 3, \ldots)$$

(3) Determination of the coefficients b_n of the cosine terms

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)\right]\sin mx\,dx$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx \qquad (m = 1,2,3,\ldots)$$

Summary of These Calculations: Fourier Coefficients, Fourier Series

Fourier Coefficients:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad (n = 1,2,\ldots)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad (n = 1,2,\ldots)$$

Fourier Series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

Ex.1) Rectangular Wave

f(x) = -k (-π<x<0) & k (0<x<π); f(x+2π)=f(x)

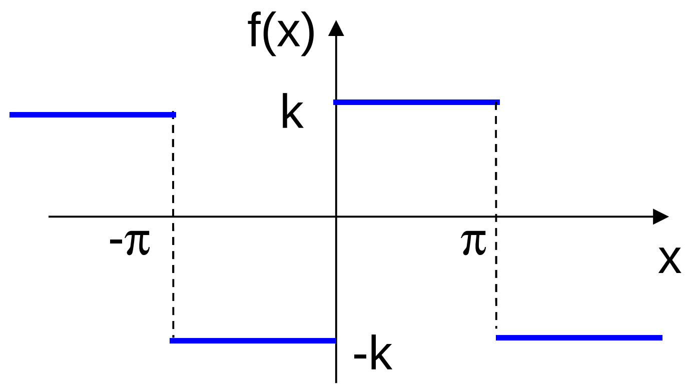
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi}(k)\cos nx\,dx\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi}(k)\sin nx\,dx\right] = \frac{2k}{n\pi}(1-\cos n\pi)$$

$$\left(b_1 = \frac{4k}{\pi}, b_2 = 0, b_3 = \frac{4k}{3\pi}, b_4 = 0, b_5 = \frac{4k}{5\pi}, \ldots\right)$$

$$\Rightarrow f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$



(See Figure. 238 for partial sums of Fourier series)

Orthogonality of the Trigonometric System

- Trigonometric system is *orthogonal* on the interval $-\pi \le x \le \pi$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad (m \neq n), \qquad \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0 \quad (m \neq n)$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0 \quad (\text{including } m = n)$$

At discontinuous point, Fourier series converge to the average, (f(x+)+f(x-))/2)

Convergence and Sum of Fourier Series

Theorem 1: A periodic function f(x) (period 2π, $-\pi \le x \le \pi$)

~ piecewise continuous

~ with a left-hand derivative and right-hand derivative at each point

→ Fourier series of f(x) is convergent. Its sum is f(x).

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx = \left.\frac{f(x)\sin nx}{n\pi}\right|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx \, dx$$

$$= \left.\frac{f'(x)\cos nx}{n^2\pi}\right|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx \, dx \qquad (|f''(x)| < M)$$

$$|a_n| = \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos nx \, dx\right| < \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M dx = \frac{2M}{n^2}, \quad |b_n| = \frac{2M}{n^2}$$

Convergent ! $\quad |f(x)| \sim |a_0| + 2M\left(1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots\right)$

10.3. Functions of Any Period p=2L

- Transition from period p=2π to period p=2L
- Function f(x) with period p=2L:

(**Trigonometric Series**)

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}\right)x + b_n \sin\left(\frac{n\pi}{L}\right)x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad (n = 1,2,\ldots)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad (n = 1,2,\ldots)$$

$$v = \frac{n\pi}{L}\ (-\pi \le v \le \pi) \Leftrightarrow x = \frac{Lv}{\pi}\ (-L \le x \le L)$$

$$f(x) \to g(v)$$

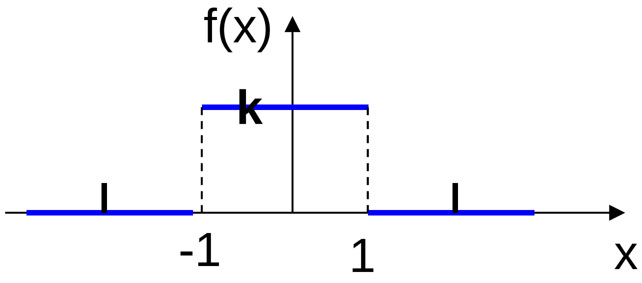
$$g(v) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(v)\,dv, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(v)\cos nv\,dv \quad (n = 1,2,\ldots)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v)\sin nv\,dv \quad (n = 1,2,\ldots)$$

Ex.1) Periodic square wave

f(x) = 0 (-2 < x < -1); k (-1 < x < 1); 0 (1 < x < 2)   p=2L=4, L=2



$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{4}\int_{-1}^{1} k\,dx = \frac{k}{2}, \quad a_n = \frac{1}{2}\int_{-1}^{1} k\cos\left(\frac{n\pi x}{2}\right)dx = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

$$\Rightarrow a_n = \frac{2k}{n\pi}\ \ (n = 1,5,9,\ldots),\ a_n = -\frac{2k}{n\pi}\ \ (n = 3,7,11,\ldots)$$

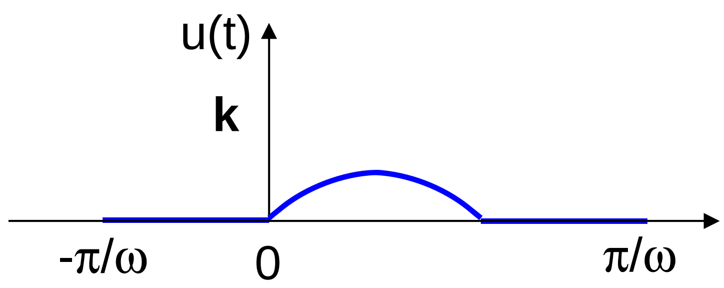
$$b_n = \frac{1}{2}\int_{-1}^{1} k\sin\left(\frac{n\pi x}{2}\right)dx = 0$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - \cdots\right)$$

Ex. 2) Half-wave rectifier

u(t) = 0 (-L < t < 0);   E$\sin\omega t$ (0 < t < L)

p = 2L = 2π/ω



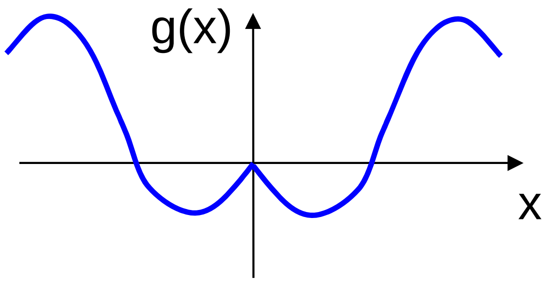
10.4. Even and Odd Functions. Half-Range Expansions

- If a function is even or odd → more compact form of Fourier series

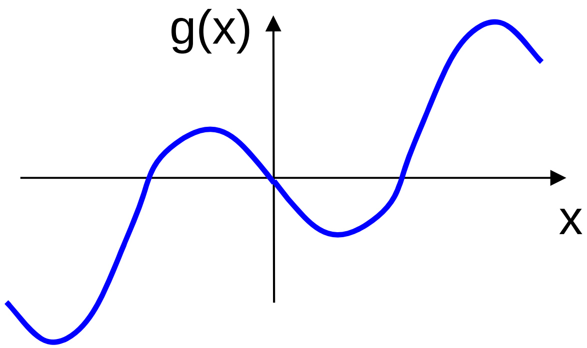
Even and Odd Functions

Even function y=g(x): g(-x) = g(x) for all x (symmetric w.r.t. y-axis)
Odd function y=g(x): g(-x) = -g(x) for all x

Even function



Odd function



Three Key Facts

(1) For even function, g(x), $\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$

(2) For odd function, h(x), $\int_{-L}^{L} h(x)dx = 0$

(3) Production of an even and an odd function → odd function

let q(x) = g(x)h(x), then q(-x) = g(-x)h(-x) = -g(x)h(x) = -q(x)

In the Fourier series,

f(x) even → f(x)sin(nπx/L) odd, then b_n=0

f(x) odd → f(x)cos(nπx/L) odd, then a_0 & a_n=0

Theorem 1: Fourier cosine series, Fourier sine series

(1) Fourier cosine series for ***even*** function with period 2L

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \left(a_0 = \frac{1}{L}\int_0^L f(x)\,dx,\ a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad (n = 1,2,\ldots)\right)$$

For even function with period 2π

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \left(a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx \quad (n = 1,2,\ldots)\right)$$

(2) Fourier sine series for ***odd*** function with period 2L

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \qquad \left(b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad (n = 1,2,\ldots)\right)$$

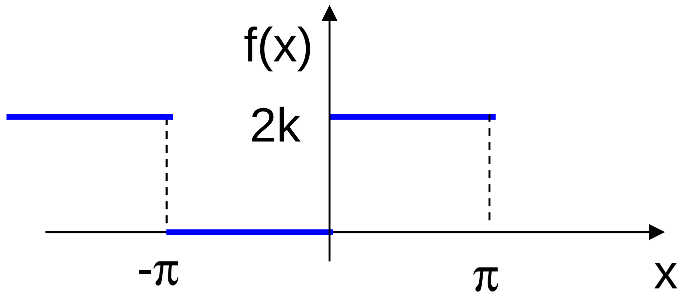
For odd function with period 2π

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \left(b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \quad (n = 1,2,\ldots)\right)$$

Theorem 2: **Sum of functions**

- Fourier coefficients of a sum of $f_1 + f_2$
  → sums of the corresponding Fourier coefficients of f_1 and f_2.
- Fourier coefficients of cf → c times the corresponding coefficients of f.

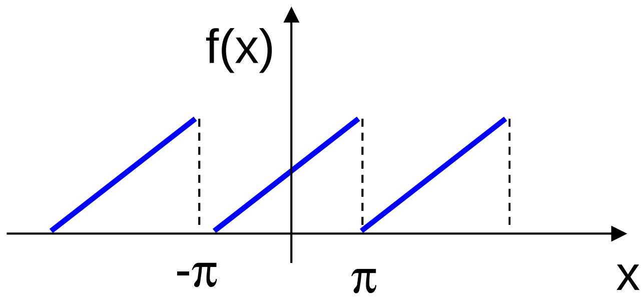
Ex. 1) Rectangular pulse



$$\Rightarrow f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right) \;\mathbf{+\,k}$$

previous result by Ex.1 in 10.2

Ex. 2) Sawtooth wave: $f(x) = x + \pi$ $(-\pi < x < \pi)$ and $f(x + \pi) = f(x)$



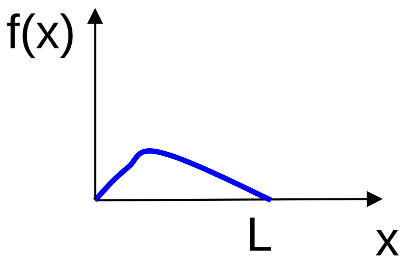
$$f = f_1 + f_2 \quad (f_1 = x,\ f_2 = \pi)$$

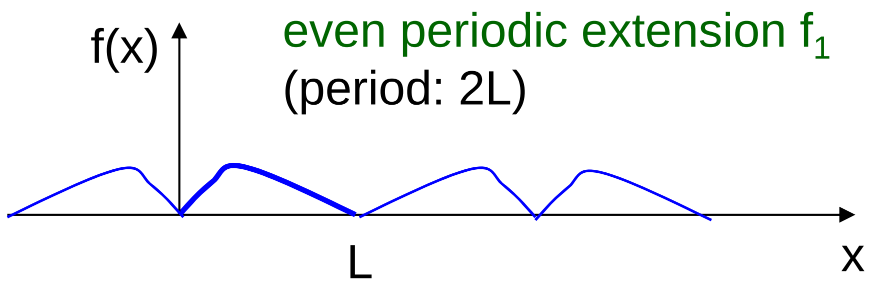
for $f_2 = \pi$: → π

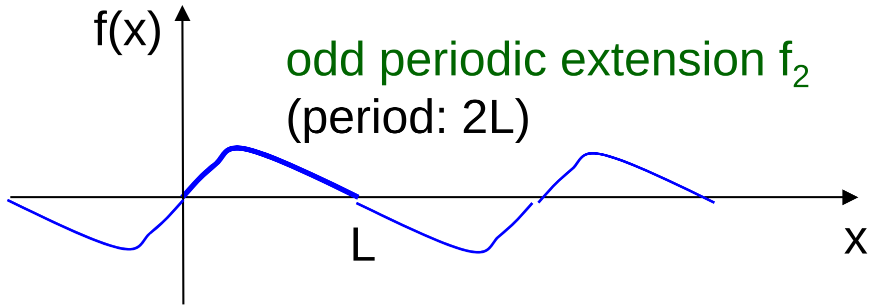
for odd $f_1 = x$ → $b_n = -\dfrac{2}{\pi}\cos n\pi$

$$\Rightarrow f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

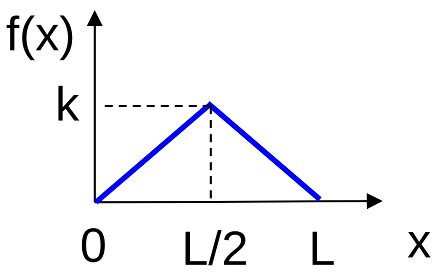
Half-Range Expansions







Ex. 1) "Triangle" and its-half-range expansions



$$f(x) = \frac{2k}{L} x \quad 0 < x < \frac{L}{2}; \quad \frac{2k}{L}(L - x) \quad \frac{L}{2} < x < L$$

(a) Even periodic extension: use $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

(b) Odd periodic extension: use $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

10.7. Approximation by Trigonometric Polynomials

- Fourier series ~ applied to *approximation theory*

$$f(x) \approx a_0 + \sum_{n=1}^{N}\left(a_n \cos nx + b_n \sin nx\right)$$

- Trigonometric polynomial of degree N:

$$F(x) = A_0 + \sum_{n=1}^{N}\left(A_n \cos nx + B_n \sin nx\right)$$

(Minimize the error by usage of the F(x) !)

- Total square error: $E = \int_{-\pi}^{\pi} (f - F)^2 dx$

- Determination of the coefficients of F(x) for minimum E

$$E = \int_{-\pi}^{\pi} f^2 dx \quad - \quad 2\int_{-\pi}^{\pi} fF dx \quad + \quad \int_{-\pi}^{\pi} F^2 dx$$

$$\int_{-\pi}^{\pi} F^2 dx = \pi\left(2A_0^2 + A_1^2 + \cdots + A_N^2 + B_1^2 + \cdots + B_N^2\right)$$

$$\left(\int_{-\pi}^{\pi} \cos^2 nx dx = \int_{-\pi}^{\pi} \sin^2 nx dx = \pi; \ \int_{-\pi}^{\pi} (\cos nx)(\sin mx) dx = 0\right)$$

$$\int_{-\pi}^{\pi} fF dx = \pi\left(2A_0 a_0 + A_1 a_1 + \cdots + A_N a_N + B_1 b_1 + \cdots + B_N b_N\right)$$

$$\left(a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, \cdots\right)$$

$$\Rightarrow E = \int_{-\pi}^{\pi} f^2 dx - 2\pi\left[2A_0 a_0 + \sum_{n=1}^{N}(A_n a_n + B_n b_n)\right] + \pi\left[2A_0^2 + \sum_{n=1}^{N}(A_n^2 + B_n^2)\right]$$

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi\left[2a_0^2 + \sum_{n=1}^{N}(a_n^0 + b_n^0)\right] \text{ when } A_n = a_n, B_n = b_n$$

$$E - E^* = \pi\left[2(A_0 - a_0)^2 + \sum_{n=1}^{N}\left((A_n - a_n)^2 + (B_n - b_n)^2\right)\right]$$

($E - E^* \geq 0$)

Theorem 1: **Minimum square error**

- Total square error, E, is minimum iff coefficients of F are the Fourier coefficients of f.
- Minimum value is E*

- From E*, better approximation as N increases

Bessel inequality: $2a_0^2 + \sum_{n=1}^{\infty}(a_n^0 + b_n^0) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx$ for any f(x)

Parseval's equality: $2a_0^2 + \sum_{n=1}^{\infty}(a_n^0 + b_n^0) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx$

Ex. 1) Square error for the sawtooth wave