9.8. Divergence Theorem. Further Applications

Ex. 1) Divergence indep. of coordinates. Invariance of divergence

- Use mean value theorem: $\iiint_T f(x,y,z)\,dxdydz = f(x_0,y_0,z_0)\,V(T)$

$$f = \underline{\nabla}\cdot\underline{F}(x_0,y_0,z_0) = \frac{1}{V(T)}\iiint_T \underline{\nabla}\cdot\underline{F}\,dV = \frac{1}{V(T)}\iint_{S(T)} \underline{F}\cdot\underline{n}\,dA$$

$$\Rightarrow \underline{\nabla}\cdot\underline{F}(x_1,y_1,z_1) = \lim_{d(T)\to 0}\frac{1}{V(T)}\iint_{S(T)} \underline{F}\cdot\underline{n}\,dA$$

Ex. 2) Physical interpretation of the divergence

Steady flow of an incompressible fluid with $\rho=1$

Total mass of fluid that flows around S from T per time: $\iint_S \underline{v}\cdot\underline{n}\,dA$

Average flow out of T: $\frac{1}{V}\iint_S \underline{v}\cdot\underline{n}\,dA$

→ For a steady incompressible flow, $\iint_S \underline{v}\cdot\underline{n}\,dA = 0 \Rightarrow \underline{\nabla}\cdot\underline{v} = 0$

Ex. 3) Heat equation: *also, see Transport Phenomena by Bird et al., (2002)*
Fundamentals of Momentum, Heat, and Mass Transfer by Welty et al. (1976, 2000)

Potential Theory. Harmonic Functions

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Potential theory, it solution → a harmonic function

Ex. 4) $$\iiint_T \underline{\nabla} \cdot \underline{\nabla} f \, dV = \iiint_T \nabla^2 f \, dV = \iint_S \nabla f \cdot n \, dA = \iint_S \frac{\partial f}{\partial n} dA$$

<u>**Theorem 1:**</u> f(x,y,z): a harmonic function in D. Above form in Ex. 4 is zero.

Ex. 5) Green's first formula:

$$\iiint_T \underline{\nabla} \cdot (f \, \underline{\nabla} g) \, dV = \iiint_T \left(\underline{\nabla} f \cdot \underline{\nabla} g + f \left(\underline{\nabla} \cdot \underline{\nabla} g\right)\right) dV = \iint_S f \, \underline{\nabla} g \cdot \underline{n} \, dA = \iint_S f \, \frac{\partial g}{\partial n} dA$$

Green's second formula:

$$\iiint_T \left(f \nabla^2 g - g \nabla^2 f\right) dV = \iint_S \left(f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} \right) dA$$

Ex. 6) $\nabla^2 f = 0$ (f : harmonic func.), $f = 0$ in S

$$\iiint_T \left(\underline{\nabla} f \cdot \underline{\nabla} f\right) dV = \iiint_T \left|\underline{\nabla} f\right|^2 dV = 0$$

<u>**Theorem 2:**</u> **Uniqueness**

From the Ex. 6, f is identically zero in T.

9.9. Stokes's Theorem

- Vector form of Green theorem: $\iint_R (\underline{\nabla} \times \underline{F}) \cdot \underline{k}\, dxdy = \oint_C \underline{F} \cdot d\underline{r} \quad (\underline{F} = F_1 \underline{i} + F_2 \underline{j})$

Theorem 1: **Stokes's Theorem** (Surface integrals ⇔ Line integrals)

S: piecewise smooth oriented surface in space

C: boundary of S

F(x,y,z) which has continuous first partial derivatives in D

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n}\, dA = \oint_C \underline{F} \cdot \underline{r}' ds \quad \left(r' = \frac{dr}{ds}, \text{ unit tangent vector} \right)$$

$$\iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) N_3 \right] dudv = \oint_{\underline{C}} (F_1 dx + F_2 dy + F_3 dz)$$

$$\iint_S \underline{F} \cdot \underline{n}\, dA = \iint_R \underline{F}(\underline{r}(u,v)) \cdot \underline{N}(u,v)\, dudv$$

Ex. 1) $\underline{F} = y\underline{i} + z\underline{j} + x\underline{k}, \; S : z = f(x,y) = 1 - (x^2 + y^2)$

(1) $\underline{r} = \cos s\, \underline{i} + \sin s\, \underline{j} \quad \Rightarrow \oint_C \underline{F} \cdot \underline{r}' d\underline{s} = \int_0^{2\pi} -\sin^2 s\, ds = -\pi$

(2) $\underline{\nabla} \times \underline{F} = -\underline{i} - \underline{j} - \underline{k}, \; \underline{N} = \underline{r}_u \times \underline{r}_v = 2x\underline{i} + 2y\underline{j} + \underline{k}$

$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot \underline{n} dA = \iint_R (-2x - 2y - 1) dxdy = -\pi$$

Polar coord.

$$\iint_R \left(\frac{\partial F_1}{\partial z} N_2 - \frac{\partial F_1}{\partial y} N_3 \right) du dv = \oint_{\underline{C}} F_1 dx$$

$$\iint_R \left(-\frac{\partial F_2}{\partial z} N_1 + \frac{\partial F_2}{\partial x} N_3 \right) du dv = \oint_{\underline{C}} F_2 dy, \quad \iint_R \left(\frac{\partial F_3}{\partial y} N_1 - \frac{\partial F_3}{\partial x} N_2 \right) du dv = \oint_{\underline{C}} F_3 dz$$

$$Z = f(x, y), \ \underline{r}(u, v) = \underline{r}(x, y) = x\underline{i} + y\underline{j} + f\underline{k}$$

$$\underline{N} = \underline{r}_u \times \underline{r}_v = \underline{r}_x \times \underline{r}_y = -f_x \underline{i} - f_y \underline{j} + \underline{k}$$

Green's theorem with $F_2=0$

$$\iint_{S^*} \left(\frac{\partial F_1}{\partial z} (-f_y) - \frac{\partial F_1}{\partial y} (1) \right) dx dy \quad = \quad \oint_{C^*} F_1 dx = \iint_{S^*} -\frac{\partial F_1}{\partial y} dx dy$$

$$\left(-\frac{\partial F_1(x, y, f(x, y))}{\partial y} = -\frac{\partial F_1(x, y, z)}{\partial y} - \frac{\partial F_1(x, y, z)}{\partial z} \frac{\partial f}{\partial y} \right)$$

Ex. 2) Green's theorem

$$\underline{F} = F_1 \underline{i} + F_2 \underline{j}, \text{ S in xy plane}$$

$$(\underline{\nabla} \times \underline{F}) \cdot \underline{n} = (\underline{\nabla} \times \underline{F}) \cdot \underline{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy = \oint_C (F_1 dx + F_2 dy)$$

Ex. 3) $\oint_C \underline{F}\cdot\underline{r}'ds\,?,\;\; C: x^2+y^2=4, z=-3,\;\; \underline{F}=y\underline{i}+xz^3\underline{j}-zy^3\underline{k}$

$$\left(\underline{\nabla}\times\underline{F}\right)\cdot\underline{n}=\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=-28\Rightarrow\oint_C \underline{F}\cdot\underline{r}'ds=-28(4\pi)$$

Ex. 4) Physical interpretation of the curl. Circulation

$$\oint_C \underline{F}\cdot\underline{r}'ds=\iint_S\left(\underline{\nabla}\times\underline{F}\right)\cdot\underline{n}\;dA=\left(\underline{\nabla}\times\underline{F}\right)\cdot\underline{n}(P^*)A\;\Rightarrow\left(\underline{\nabla}\times\underline{F}\right)\cdot\underline{n}(P^*)=\frac{1}{A}\oint_C \underline{F}\cdot\underline{r}'ds$$

$\underline{F}=\underline{v}$, the circulation of the flow around C: $\oint_C \underline{v}\cdot\underline{r}'ds$

$$\left(\underline{\nabla}\times\underline{v}\right)\cdot\underline{n}(P)=\lim_{r\to 0}\frac{1}{A}\oint_C \underline{v}\cdot\underline{r}'ds$$

Stokes's Theorem Applied to Path Independence

$$\oint_C\left(F_1dx+F_2dy+F_3dz\right)=\oint_C \underline{F}\cdot\underline{r}'ds=\iint_S\left(\underline{\nabla}\times\underline{F}\right)\cdot\underline{n}dA=0$$

for path independence $\left(\underline{\nabla}\times\underline{F}=\underline{0}\right)$