Further Applications of Green's Theorem

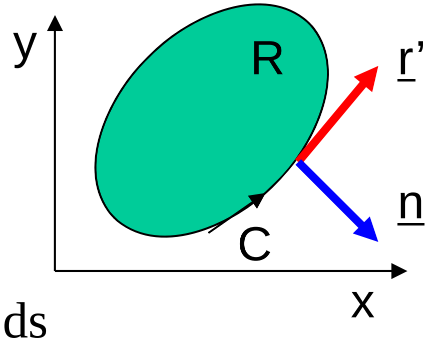
Ex. 2) Area of a plane region

$$F_1 = 0, F_2 = x \Rightarrow \iint_R dxdy = \oint_C xdy; \quad F_1 = -y, F_2 = 0 \Rightarrow \iint_R dxdy = -\oint_C ydy$$

$$A = \frac{1}{2}\oint_C (xdy - ydx)$$

-for ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ x = a\cos t, y = b\sin t \Rightarrow A = \frac{1}{2}\int_0^{2\pi}(xy' - yx')dt = \pi ab$

-in polar coordinates: $x = r\cos\theta, y = r\sin\theta \Rightarrow A = \frac{1}{2}\oint_C r^2 d\theta$



Ex. 4) Double integral of the Laplacian of a function

$$w = w(x,y), F_1 = -\frac{\partial w}{\partial y}, F_2 = \frac{\partial w}{\partial x} \Rightarrow \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \nabla^2 w$$

$$\oint_C (F_1 dx + F_2 dy) = \oint_C (F_1 \frac{dx}{ds} + F_2 \frac{du}{ds})ds = \oint_C (-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds})ds$$

$$\left(\underline{\nabla} w = \frac{\partial w}{\partial x}\underline{i} + \frac{\partial w}{\partial y}\underline{j} \quad \& \quad \underline{n} = \frac{\partial y}{\partial s}\underline{i} - \frac{\partial x}{\partial s}\underline{j}\right)$$

$$\iint_R \nabla^2 w \, dxdy = \oint_C \underline{\nabla} w \cdot \underline{n} \, ds = \oint_C \frac{\partial w}{\partial n} ds$$

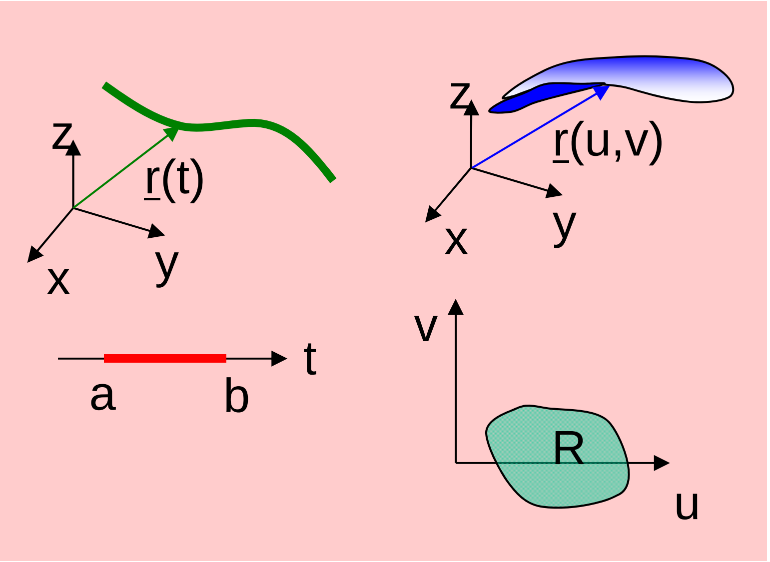
s: arc length of C

$\underline{n}$: unit normal vector to C perpendicular to unit tangent $\underline{r}'$

9.5. Surfaces for Surface Integrals

Representations of Surfaces

Surface S: z=f(x,y) or g(x,y,z)=0



<u>Parametric representations</u>:

$$\underline{r}(u,v) = x(u,v)\underline{i} + y(u,v)\underline{j} + z(u,v)\underline{k}; \quad (u,v) \text{ in } R$$

Ex. 1) Circular cylinder

$$x^2 + y^2 = a^2, -1 \le z \le 1$$

$$\underline{r}(u,v) = a\cos u\underline{i} + a\sin u\underline{j} + v\underline{k}, \quad 0 \le u \le 2\pi, -1 \le v \le 1$$

Ex. 2) Sphere

$$x^2 + y^2 + z^2 = a^2$$

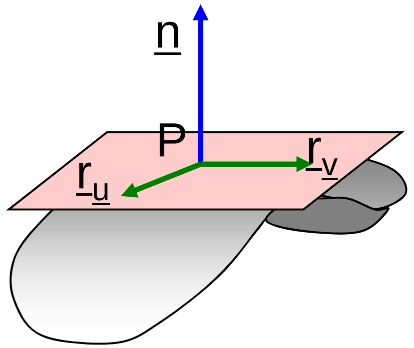
$$\underline{r}(u,v) = a\cos v\cos u\underline{i} + a\cos v\sin u\underline{j} + a\sin v\underline{k}, \quad 0 \le u \le 2\pi, -\pi/2 \le v \le \pi/2$$

Ex. 3) Circular cone

$$z = \sqrt{x^2 + y^2}, 0 \le z \le H$$

$$\underline{r}(u,v) = u\cos v\underline{i} + u\sin v\underline{j} + u\underline{k}, \quad 0 \le u \le H, 0 \le v \le 2\pi$$

Tangent Plane and Surface Normal



- Normal vector of S at point P ⊥ tangent vector of S at P

$$\underline{\tilde{r}}(t) = \underline{r}(u(t), v(t))$$

$$\underline{\tilde{r}}' = \frac{d\underline{\tilde{r}}}{dt} = \frac{\partial \underline{r}}{\partial u}u' + \frac{\partial \underline{r}}{\partial v}v' = \underline{r}_u u' + \underline{r}_v v'$$

($\underline{r}_u$' and $\underline{r}_u$': tangent to S at P)

- Normal vector $\underline{N}$ of S at P: $\underline{N} = \underline{r}_u \times \underline{r}_v \neq \underline{0}$

- Unit normal vector $\underline{n}$: $\underline{n} = \frac{1}{|\underline{N}|}\underline{N} = \frac{1}{|\underline{r}_u \times \underline{r}_v|}\underline{r}_u \times \underline{r}_v$ or $\frac{1}{|\nabla g|}\underline{\nabla g}$ $(g(x,y,z)=0)$

Ex. 4) Unit normal vector of a sphere

$$g(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0 \Rightarrow \underline{n}(x,y,z) = \frac{x}{a}\underline{i} + \frac{y}{a}\underline{j} + \frac{z}{a}\underline{k}$$

9.6. Surface Integrals

- Surface integral over S: $\iint_S \underline{F}\cdot\underline{n}\, dA = \iint_R \underline{F}(\underline{r}(u,v))\cdot\underline{N}(u,v)\, du dv$

Normal component of $\underline{F}$

$$\left(\underline{n}\, dA = \underline{n}|\underline{N}|\, du dv = \underline{N}\, du dv\right)$$

(e.g., $\underline{F}$=ρ$\underline{v}$: mass flux across S) → “Flux integral”

$$\underline{F} = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$$

$$\underline{n} = \cos\alpha\,\underline{i} + \cos\beta\,\underline{j} + \cos\gamma\,\underline{k} \qquad (\cos\alpha = \underline{n}\cdot\underline{i}, \ldots)$$

$$\underline{N} = N_1\underline{i} + N_2\underline{j} + N_3\underline{k}$$

$$\iint_S \underline{F}\cdot\underline{n}\,dA = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\,dA$$

$$= \iint_R (F_1N_1 + F_2N_2 + F_3N_3)\,du\,dv$$

Ex. 1) Parabolic cylinder S: $y=x^2$, $0 \le x \le 2$, $0 \le z \le 3$

$\underline{v} = \underline{F} = (3z^2, 6, 6xz)$

$$S: \underline{r} = (u, u^2, v) \qquad (0 \le u \le 2, 0 \le v \le 3)$$

$$\underline{r}_u = (1,2u,0),\ \underline{r}_v = (0,0,1),\ \underline{N} = \underline{r}_u \times \underline{r}_v = (2u,-1,0)$$

$$\int_0^3\int_0^2 \underline{F}\cdot\underline{N}\,du\,dv = \int_0^3\int_0^2 (6uv^2 - 6)\,du\,dv = 72$$

- Integral depends on the choice of the unit normal vector $\underline{n}$.
  → Integral over an oriented surface S

Theorem 1: **Change of orientation**

The replacement of $\underline{n}$ by $-\underline{n}$ corresponds to the multiplication of the integral by –1.

Another Way of Writing Surface Integrals

$$\iint_S F_1 \cos\alpha \, dA = \iint_S F_1 dydz; \quad \iint_S F_2 \cos\beta \, dA = \iint_S F_2 dzdx; \quad \iint_S F_3 \cos\gamma \, dA = \iint_S F_3 dxdy$$

$$\Rightarrow \iint_S \underline{F} \cdot \underline{n} \, dA = \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy\right)$$

$$\iint_S F_3 \cos\gamma \, dA = +\iint_R F_3(x, y, h(x, y)) \, dxdy \quad \text{for } \cos\gamma > 0, \ z = h(x, y)$$

$$\iint_S F_3 \cos\gamma \, dA = -\iint_R F_3(x, y, h(x, y)) \, dxdy \quad \text{for } \cos\gamma < 0, \ z = h(x, y)$$

Ex. 4) Parabolic cylinder S: $y=x^2$, $0 \le x \le 2$, $0 \le z \le 3$, $\underline{v} = \underline{F} = (3z^2, 6, 6xz)$

$$S: \underline{r} = (u, u^2, v) \qquad (0 \le u \le 2, 0 \le v \le 3)$$

$$\underline{r}_u = (1,2u,0), \ \underline{r}_v = (0,0,1), \ \underline{N} = \underline{r}_u \times \underline{r}_v = (2u,-1,0) = (2x,-1,0)$$

$$\int_0^3 \int_0^4 3z^2 dydz - \int_0^2 \int_0^3 6dzdx = 72$$

Surface Integrals Without Regard to Orientation

$$\iint_S G(\underline{r}) dA = \iint_R G(\underline{r}(u, v)) \, |\underline{N}(u, v)| dudv$$

$$\left(dA = |\underline{N}| \, dudv\right)$$

Area of A(S) of S:

$$A(S) = \iint_S dA = \iint_R |\underline{r}_u \times \underline{r}_v| dudv$$

Ex. 5~7)

Representation z=f(x,y). S: z=f(x,y) → r = (x,y,z) = (u,v,f)

$$|\underline{N}| = |\underline{r}_u \times \underline{r}_v| = |[-f_u, -f_v, 1]| = \sqrt{1 + f_u^2 + f_v^2}$$

$$\iint_S G(\underline{r})\,dA = \iint_{R^*} G(x, y, f(x, y)) \left| \sqrt{1 + (\partial f/\partial x)^2 + (\partial f/\partial y)^2} \right| dxdy$$

$$A(S) = \iint_{R^*} \left| \sqrt{1 + (\partial f/\partial x)^2 + (\partial f/\partial y)^2} \right| dxdy$$

(R*: projection of S into xy plane)