Chap. 7. Linear Algebra: Matrix Eigenvalue Problems

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$$

square matrix

unknown vector

unknown scalar

$\underline{x} = \underline{0}$: (no practical interest)

$\underline{x} \neq \underline{0}$: eigenvectors of **A**; exist only for certain values of λ (eigenvalues or characteristic roots)

→ Multiplication of **A** = same effect as the multiplication of **x** by a scalar λ

→ Important to determine the stability of chemical & biological processes

- Eigenvalue: special set of scalars associated with a linear systems of equations. Each eigenvalue is paired with a corresponding eigenvectors.

7.1. Eigenvalues, Eigenvectors

- Eigenvalue problems: $\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$ or $\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)\underline{x} = \underline{0}$

eigenvectors

eigenvectors

Set of eigenvalues: spectrum of A

How to Find Eigenvalues and Eigenvectors

Ex. 1.)

$$\underline{\underline{A}} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \qquad \begin{aligned} -5x_1 + 2x_2 &= \lambda x_1 \\ 2x_1 - 2x_2 &= \lambda x_2 \end{aligned} \qquad (\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}$$

In homogeneous linear system, nontrivial solutions exist when det (**A**-λ**I**)=0.

Characteristic equation of **A**:

$$D(\lambda) = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 + 7\lambda + 6 = 0$$

Characteristic polynomial

Characteristic determinant

Eigenvalues: λ_1=-1 and λ_2= -6

Eigenvectors: for λ_1=-1, $\underline{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for λ_2=-6, $\underline{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

obtained from Gauss elimination

General Case

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \lambda x_1 \\ &\vdots \\ &a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = \lambda x_n \end{aligned}$$

$$\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)\underline{x} = \underline{0}, \quad D(\lambda) = \det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) = 0$$

Theorem 1:
Eigenvalues of a square matrix **A** → roots of the characteristic equation of **A**.
nxn matrix has at least one eigenvalue, and at most n numerically different eigenvalues.

Theorem 2:
If $\underline{x}$ is an eigenvector of a matrix **A**, corresponding to an eigenvalue λ,
so is $k\underline{x}$ with any $k \neq 0$.

Ex. 2) multiple eigenvalue

- Algebraic multiplicity of λ: order M_λ of an eigenvalue λ
  Geometric multiplicity of λ: number of m_λ of linear independent eigenvectors corresponding to λ. (=dimension of eigenspace of λ)

  In general, $m_\lambda \leq M_\lambda$

  Defect of λ: $\Delta_\lambda = M_\lambda - m_\lambda$

Ex 3) algebraic & geometric multiplicity, positive defect

Ex. 4) complex eigenvalues and eigenvectors

7.2. Some Applications of Eigenvalue Problems

Ex. 1) Stretching of an elastic membrane.

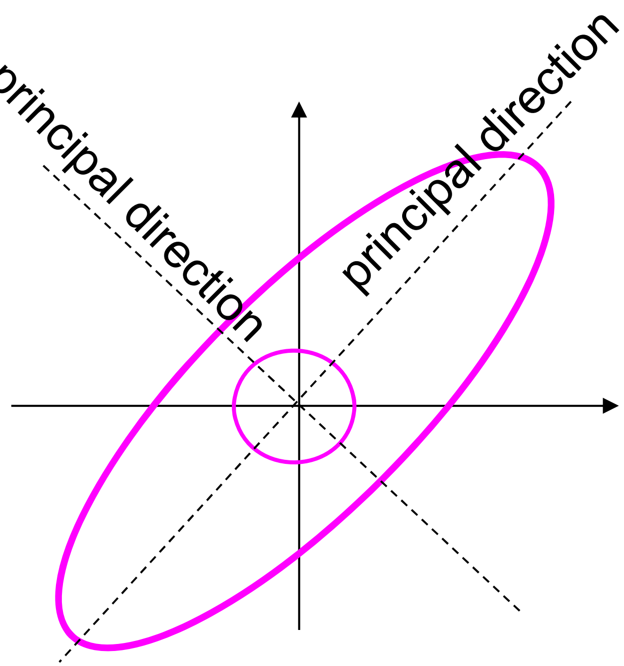
Find the principal directions: direction of position vector $\underline{x}$ of P

= (same or opposite) direction of the position vector $\underline{y}$ of Q

$$x_1^2 + x_2^2 = 1, \quad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underline{y} = \underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \quad \Rightarrow \lambda_1 = 8, \ \underline{x}_1 \text{ for } \lambda_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2, \ \underline{x}_2 \text{ for } \lambda_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$



Eigenvalue represents speed of response

Eigenvector ~ direction

Ex. 4) Vibrating system of two masses on two springs

$$y_1'' = -5y_1 + 2y_2$$

$$y_2'' = 2y_1 - 2y_2$$

Solution vector: $\underline{y} = \underline{x}e^{wt}$

$\Rightarrow \underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \quad (\lambda = w^2)$ solve eigenvalues and eigenvectors

$$\Rightarrow \underline{y} = \underline{x}_1(a_1 \cos t + b_1 \sin t) + \underline{x}_2(a_2 \cos\sqrt{6}t + b_2 \sin\sqrt{6}t)$$

Examples for stability analysis of linear ODE systems using eigenmodes

Stability criterion: signs of real part of eigenvalues of the matrix

$$\dot{x} = \frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x}$$

A determine the stability of the linear system.
Re(λ) < 0: stable
Re(λ) > 0: unstable

Ex. 1) Node-sink

$$\dot{x}_1 = -0.5x_1 + x_2 \quad \Rightarrow \lambda_1 = -0.5$$

$$\dot{x}_2 = -2x_2 \qquad\qquad \lambda_2 = -2$$

stable

Ex. 2) Saddle

$$\dot{x}_1 = 2x_1 + x_2 \qquad \Rightarrow \lambda_1 = -1.5616,\ \underline{x}_1 \text{ for } \lambda_1 = \begin{pmatrix} 0.2703 \\ -0.9628 \end{pmatrix}$$

$$\dot{x}_2 = 2x_1 - x_2$$

unstable

$$\lambda_2 = 2.5616,\ \underline{x}_2 \text{ for } \lambda_2 = \begin{pmatrix} 0.8719 \\ 0.4896 \end{pmatrix}$$

Phase plane ?

Ex. 3) Unstable focus

$$\dot{x}_1 = x_1 + 2x_2 \qquad \Rightarrow \lambda_1 = 1 + 2i$$

$$\dot{x}_2 = -2x_1 + x_2 \qquad \lambda_2 = 1 - 2i$$

unstable

Phase plane ?

Ex. 4) Center

$$\dot{x}_1 = -x_1 - x_2 \qquad \Rightarrow \lambda_1 = 0 + 1.7321i$$

$$\dot{x}_2 = 4x_1 + x_2 \qquad \lambda_2 = 0 - 1.7321i$$

7.3. Symmetric, Skew-Symmetric, and Orthogonal Matrices

- Three classes of real square matrices

(1) Symmetric:

$$\underline{\underline{A}}^T = \underline{\underline{A}}, \quad a_{kj} = a_{jk}, \quad \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

(2) Skew-symmetric:

$$\underline{\underline{A}}^T = -\underline{\underline{A}}, \quad a_{kj} = -a_{jk}, \quad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix} \quad \textit{Zero-diagonal terms}$$

(3) Orthogonal:

$$\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}, \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\underline{\underline{A}} = \underline{\underline{R}} + \underline{\underline{S}},$$

$$\underline{\underline{R}} = \frac{1}{2}\left(\underline{\underline{A}} + \underline{\underline{A}}^T\right) \text{ symmetric}$$

$$\underline{\underline{S}} = \frac{1}{2}\left(\underline{\underline{A}} - \underline{\underline{A}}^T\right) \text{ skew-symmetric}$$

Theorem 1:

(a) The eigenvalues of a symmetric matrix are real.

(b) The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.

$$\underline{\underline{A}}\,\underline{k} = \lambda \underline{k} \quad \text{Conjugate}: \overline{\underline{\underline{A}}\,\underline{k}} = \overline{\lambda \underline{k}} \Rightarrow \underline{\underline{A}}\,\overline{\underline{k}} = \overline{\lambda}\,\overline{\underline{k}} \, (\underline{\underline{A}} = \overline{\underline{\underline{A}}}, \text{ real})$$

Transpose, and then multiply $\underline{k}$: $\overline{\underline{k}}^T \underline{\underline{A}}^T \underline{k} = \overline{\underline{k}}^T \overline{\lambda}\,\underline{k} \Rightarrow \overline{\underline{k}}^T \lambda \underline{k} = \overline{\underline{k}}^T \overline{\lambda}\,\underline{k} \Rightarrow \lambda\,\overline{\underline{k}}^T \underline{k} = \overline{\lambda}\,\overline{\underline{k}}^T \underline{k}$

Ex. 3) $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ $\lambda = 2, 8$ $\begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$ $\lambda = 0, \pm 25i$

Orthogonal Transformations and Matrices

$\underline{y} = \underline{\underline{A}}\,\underline{x}$ (A : orthogonal matrix) Ex) $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

- Orthogonal transformation in the 2D plane and 3D space: ***rotation***

Theorem 2: (Invariance of inner product)

An orthogonal transformation preserves the value of the inner product of vectors.

$$\underline{u} = \underline{\underline{A}}\,\underline{a},\ \underline{v} = \underline{\underline{A}}\,\underline{b} \quad (\underline{\underline{A}} : \text{orthogonal})$$

$$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = (\underline{\underline{A}}\,\underline{a})^T (\underline{\underline{A}}\,\underline{b}) = \underline{a}^T \underline{\underline{A}}^T \underline{\underline{A}}\,\underline{b}$$

$$= \underline{a}^T (\underline{\underline{A}}^{-1} \underline{\underline{A}})\underline{b} = \underline{a}^T \underline{b} = \underline{a} \cdot \underline{b}$$

$$\underline{a} \cdot \underline{b} = \underline{a}^T \underline{b} \quad (\underline{a}, \underline{b} : \text{column vectors})$$

the length or norm of a vector in R^n given by

$$\|\underline{a}\| = \sqrt{\underline{a} \cdot \underline{a}} = \sqrt{\underline{a}^T \underline{a}}$$

Theorem 3: (Orthonormality of column and row vectors)

A real square matrix is **orthogonal** iff its column (or row) vectors, $\underline{a}^1, \ldots, \underline{a}^n$ form an **orthonormal** system

$$\underline{a}_j \cdot \underline{a}_k = \underline{a}_j^T \underline{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

$$\underline{\underline{A}}^{-1} \underline{\underline{A}} = \underline{\underline{I}} = \underline{\underline{A}}^T \underline{\underline{A}}, \begin{pmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_n^T \end{pmatrix} (\underline{a}_1 \quad \cdots \quad \underline{a}_n)$$

Theorem 4: The determinant of an orthogonal matrix has the value of +1 or –1.

$$1 = \det \underline{\underline{I}} = \det(\underline{\underline{A}}\,\underline{\underline{A}}^{-1}) = \det(\underline{\underline{A}}\,\underline{\underline{A}}^{T}) = \det \underline{\underline{A}} \det \underline{\underline{A}}^{T} = (\det \underline{\underline{A}})^2$$

Theorem 5: Eigenvalues of an orthogonal matrix A are real or complex conjugates in pairs and have absolute value 1.

7.4. Complex Matrices: Hermitian, Skew-Hermitian, Unitary

- Conjugate matrix: $\overline{\underline{\underline{A}}} = \bar{a}_{jk}, \quad \overline{\underline{\underline{A}}}^{T} = \bar{a}_{kj}$ $\qquad \underline{\underline{A}} = \begin{pmatrix} 3+4i & -5i \\ -7 & 6-2i \end{pmatrix} \Rightarrow \overline{\underline{\underline{A}}}^{T} = \begin{pmatrix} 3-4i & -7 \\ 5i & 6+2i \end{pmatrix}$

- Three classes of complex square matrices:

(1) Hermitian:

$$\overline{\underline{\underline{A}}}^{T} = \underline{\underline{A}}, \quad \bar{a}_{kj} = a_{jk}, \quad \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$$

Diagonal-terms: real
$\bar{a}_{jj} = a_{jj}$

(2) Skew-Hermitian:

$$\overline{\underline{\underline{A}}}^{T} = -\underline{\underline{A}}, \quad \bar{a}_{kj} = -a_{jk}, \quad \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$$

Diagonal-terms:
pure imag. or 0

$\bar{a}_{jj} = -a_{jj}$

(3) Unitary:

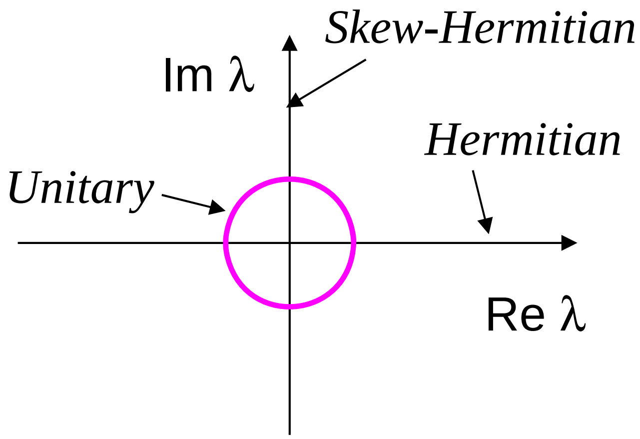
$$\overline{\underline{\underline{A}}}^{T} = \underline{\underline{A}}^{-1}, \quad \begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$$

- Generalization of section 7.3

Hermitian matrix: real → symmetric $\quad \overline{\underline{\underline{A}}}^{T} = \underline{\underline{A}}^{T} = \underline{\underline{A}}$

Skew-Hermitian matrix: real → skew-symmetric $\quad \overline{\underline{\underline{A}}}^{T} = \underline{\underline{A}}^{T} = -\underline{\underline{A}}$

Unitary matrix: real → orthogonal $\quad \overline{\underline{\underline{A}}}^{T} = \underline{\underline{A}}^{T} = \underline{\underline{A}}^{-1}$



Eigenvalues

Theorem 1:

(a) Eigenvalues of Hermitian (symmetric) matrix → real

(b) Skew-Hermitian (skew-symmetric) matrix → pure imag. or zero

(c) Unitary (orthogonal) matrix → absolute value 1

Forms

$\overline{\underline{x}}^{T}\underline{\underline{A}}\,\underline{x}$: a form in the components $x_1, \ldots, x_n$ of $\underline{x}$, **A** coefficient matrix

$$\overline{\underline{x}}^{T}\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} \overline{x}_1 & \overline{x}_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}\overline{x}_1 x_1 + a_{12}\overline{x}_1 x_2 + a_{21}\overline{x}_2 x_1 + a_{22}\overline{x}_2 x_2$$

Proof of Theorem 1:

(a) Eigenvalues of Hermitian (symmetric) matrix → real

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \quad \Rightarrow \quad \underline{\bar{x}}^T \underline{\underline{A}}\,\underline{x} = \underline{\bar{x}}^T \lambda \underline{x} = \lambda \underline{\bar{x}}^T \underline{x}$$

$$\lambda = \frac{\underline{\bar{x}}^T \underline{\underline{A}}\,\underline{x}}{\underline{\bar{x}}^T \underline{x}} = \frac{\text{real}\,?}{\text{real}\,!} \quad (\text{use } \overline{\underline{\underline{A}}}^T = \underline{\underline{A}},\ \underline{\underline{A}}^T = \overline{\underline{\underline{A}}})$$

$$\underline{\bar{x}}^T \underline{\underline{A}}\,\underline{x} = \left(\underline{\bar{x}}^T \underline{\underline{A}}\,\underline{x}\right)^T = \underline{x}^T \underline{\underline{A}}^T \underline{\bar{x}} = \underline{x}^T \overline{\underline{\underline{A}}}\,\underline{\bar{x}} = \overline{\left(\underline{\bar{x}}^T \underline{\underline{A}}\,\underline{x}\right)}$$

(b) Eigenvalues of Skew-Hermitian (skew-symmetric) matrix → pure imag. or zero

$$\underline{\bar{x}}^T \underline{\underline{A}}\,\underline{x} = -\overline{\left(\underline{\bar{x}}^T \underline{\underline{A}}\,\underline{x}\right)}$$

(c) Eigenvalues of Unitary (orthogonal) matrix → absolute value 1

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \quad \Rightarrow \text{conjugate transpose}: \left(\overline{\underline{\underline{A}}}\,\underline{\bar{x}}\right)^T = \left(\bar{\lambda}\,\underline{\bar{x}}\right)^T$$

$$\left(\overline{\underline{\underline{A}}}\,\underline{\bar{x}}\right)^T \left(\underline{\underline{A}}\,\underline{x}\right) = \left(\bar{\lambda}\,\underline{\bar{x}}\right)^T (\lambda \underline{x}) = \bar{\lambda} \lambda \underline{\bar{x}}^T \underline{x} = |\lambda|^2 \underline{\bar{x}}^T \underline{x} = \underline{\bar{x}}^T \underline{x}$$

- For general n,

$$\underline{\bar{x}}^T \underline{\underline{A}}\,\underline{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}\bar{x}_j x_k = a_{11}\bar{x}_1 x_1 + \cdots + a_{1n}\bar{x}_1 x_n$$

$$+ a_{21}\bar{x}_2 x_1 + \cdots + a_{2n}\bar{x}_2 x_n$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ a_{n1}\bar{x}_n x_1 + \cdots + a_{nn}\bar{x}_n x_n$$

- For real **A**, $\underline{x}$,

$$\underline{x}^T \underline{\underline{A}}\,\underline{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk} x_j x_k = a_{11}x_1^2 + a_{12}x_1 x_2 + \cdots + a_{1n}x_1 x_n$$

$$+ a_{21}x_2 x_1 + a_{22}x_2^2 \cdots + a_{2n}x_2 x_n$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ a_{n1}x_n x_1 + a_{n2}x_n x_2 \cdots + a_{nn}x_n^2$$

Quadratic form

- Hermitian **A**: Hermitian form,   Skew-Hermitian **A**: Skew-Hermitian form

Theorem 1: For every choice of the vector $\underline{x}$, the value of a Hermitian form is real, and the value of a skew-Hermitian form is pure imaginary or 0.

Properties of Unitary Matrices. Complex Vector Space C^n.

- Complex vector space: C^n

Inner product: $\underline{a} \cdot \underline{b} = \overline{\underline{a}}^T \underline{b}$

length or norm: $\|\underline{\underline{a}}\| = \sqrt{\underline{a} \cdot \underline{a}} = \sqrt{\overline{\underline{a}}^T \underline{a}} = \sqrt{|a_1|^2 + \cdots + |a_n|^2}$

Theorem 2: A unitary transformation, $\underline{y}=\mathbf{A}\underline{x}$ ($\mathbf{A}$: unitary matrix) preserves the value of the inner product and norm.

$$\underline{u} \cdot \underline{v} = \overline{\underline{u}}^T \underline{v} = (\overline{\underline{\underline{A}}\underline{a}})^T (\underline{\underline{A}}\underline{b}) = \overline{\underline{a}}^T \overline{\underline{\underline{A}}}^T \underline{\underline{A}}\underline{b} = \overline{\underline{a}}^T \underline{b} = \underline{a} \cdot \underline{b}$$

- Unitary system: complex analog of an orthonormal system of real vectors

$$\underline{a}_j \cdot \underline{a}_k = \overline{\underline{a}}_j^T \underline{a}_k = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

Theorem 3: A square matrix is unitary iff its column vectors form a unitary system.

Theorem 4: The determinant of a unitary matrix has absolute value 1.

$$1 = \det \underline{\underline{I}} = \det(\underline{\underline{A}}\underline{\underline{A}}^{-1}) = \det(\underline{\underline{A}}\overline{\underline{\underline{A}}}^T) = \det \underline{\underline{A}} \det \overline{\underline{\underline{A}}}^T = \det \underline{\underline{A}} \det \overline{\underline{\underline{A}}}$$
$$= \det \underline{\underline{A}} \overline{\det \underline{\underline{A}}} = |\det A|^2$$

7.5. Similarity of Matrices, Basis of Eigenvectors, Diagonalization

-Eigenvectors of n x n matrix **A** forming a basis for R^n or C^n ~ used for diagonalizing **A**

Similarity of Matrices

- n x n matirx $\underline{\underline{\hat{A}}}$ is ***similar*** to an n x n matrix **A** if $\underline{\underline{\hat{A}}} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}$ for nonsingular n x n **P**

Similarity transformation: $\underline{\underline{\hat{A}}}$ from $\underline{\underline{A}}$

Theorem 1: $\underline{\underline{\hat{A}}}$ has the same eigenvalues as **A** if $\underline{\underline{\hat{A}}}$ is similar to **A**.
$\underline{y} = \underline{\underline{P}}^{-1}\underline{x}$ is an eigenvector of $\underline{\underline{\hat{A}}}$ corresponding to the same eigenvalue,
if $\underline{x}$ is an eigenvector of **A**.

$$\underline{\underline{A}}\,\underline{x} = \lambda\underline{x} \;\Rightarrow \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{x} = \lambda\underline{\underline{P}}^{-1}\underline{x}$$

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{x} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{I}}\,\underline{x} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}\,\underline{\underline{P}}^{-1}\underline{x} = \underline{\underline{\hat{A}}}\left(\underline{\underline{P}}^{-1}\underline{x}\right) = \lambda\left(\underline{\underline{P}}^{-1}\underline{x}\right)$$

Properties of Eigenvectors

Theorem 2: $\lambda_1, \lambda_2, \ldots, \lambda_n$: distinct eigenvalues of an n x n matrix.
Corresponding eigenvectors $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n$ → a linearly independent set.

Theorem 3: n x n matrix **A** has n distinct eigenvalues → **A** has a basis of eigenvector for C^n (or R^n).

Ex. 1) $\underline{\underline{A}} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ a basis of eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Theorem 4: A Hermitian, skew-Hermitian, or unitary matrix has a basis of eigenvectors for C^n that is a *unitary* system.
A symmetric matrix has an *orthonomal* basis of eigenvectors for R^n.

$^{(1)}\underline{\underline{A}}\underline{x}_1 = \lambda_1 \underline{x}_1, \quad ^{(2)}\underline{\underline{A}}\underline{x}_2 = \lambda_2 \underline{x}_2; \quad \text{show } \underline{x}_1 \cdot \underline{x}_2 = \underline{x}_1^T \underline{x}_2 = 0$

(1) Transpose, then multiply $\underline{x}_2$ on the right: $\underline{x}_1^T \underline{\underline{A}}^T = \underline{x}_1^T \lambda_1 \rightarrow \underline{x}_1^T \underline{\underline{A}}^T \underline{x}_2 = \underline{x}_1^T \lambda_1 \underline{x}_2 = \lambda_1 \underline{x}_1^T \underline{x}_2$

(2) Multiply $\underline{x}_1^T$ on the left: $\underline{x}_1^T \underline{\underline{A}} \underline{x}_2 = \underline{x}_1^T \lambda_2 \underline{x}_2 = \lambda_2 \underline{x}_1^T \underline{x}_2$

$\lambda_1 \underline{x}_1^T \underline{x}_2 = \lambda_2 \underline{x}_1^T \underline{x}_2 \Rightarrow 0 = (\lambda_1 - \lambda_2) \underline{x}_1^T \underline{x}_2, \lambda_1 \neq \lambda_2$

Ex. 3) From Ex. 1, orthonormal basis of eigenvectors $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

- Basis of eigenvectors of a matrix **A**: useful in <u>transformation and diagonalization</u>

$$\underline{y} = \underline{\underline{A}}\underline{x}, \quad \underline{x} = c_1 \underline{x}_1 + c_2 \underline{x}_2 + \cdots + c_n \underline{x}_n \quad (\underline{x}_1, ..., \underline{x}_n : \text{basis})$$

$$\Rightarrow \underline{y} = \underline{\underline{A}} \left(c_1 \underline{x}_1 + c_2 \underline{x}_2 + \cdots + c_n \underline{x}_n \right)$$

$$= c_1 \lambda_1 \underline{x}_1 + \cdots + c_n \lambda_n \underline{x}_n$$

*Complicated calculation of **A** on $\underline{x}$ → sum of simple evaluation on the eigenvectors of **A**.*

Diagonalization

Theorem 5: If an n x n matrix **A** has a basis of eigenvectors, then

$$\underline{\underline{D}} = \underline{\underline{X}}^{-1}\underline{\underline{A}}\,\underline{\underline{X}}$$

is diagonal, with the eigenvalues of **A** on the main diagonal.
(**X**: matrix with eigenvectors as column vectors)

$$\underline{\underline{D}}^{m} = \underline{\underline{X}}^{-1}\underline{\underline{A}}^{m}\underline{\underline{X}} \quad (m = 2, 3, \ldots)$$

→ n x n matrix **A** is diagonalizable iff **A** has n linearly independent eigenvectors.
→ **Sufficient condition for diagonalization**: If an n x n matrix **A** has n distinct eigenvalues, it is diagonalizable.

Ex. 4) $\underline{\underline{A}} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$; eigenvectors $\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$; $\underline{\underline{X}} = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$; $\underline{\underline{X}}^{-1}\underline{\underline{A}}\,\underline{\underline{X}} = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}$

$\underline{x}_1, \ldots, \underline{x}_n$: basis of eigenvectors of **A** for C^n (or R^n) corresponding to $\lambda_1, \ldots, \lambda_n$

$$\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{A}}\left[\underline{x}_1 \;\; \cdots \;\; \underline{x}_n\right] = \left[\underline{\underline{A}}\,\underline{x}_1 \;\; \cdots \;\; \underline{\underline{A}}\,\underline{x}_n\right] = \left[\lambda_1 \underline{x}_1 \;\; \cdots \;\; \lambda_n \underline{x}_n\right] = \underline{\underline{X}}\,\underline{\underline{D}}$$

$$\Rightarrow \underline{\underline{X}}^{-1}\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{D}}$$

$$\underline{\underline{D}}^{2} = \underline{\underline{X}}^{-1}\underline{\underline{A}}\,\underline{\underline{X}}\,\underline{\underline{X}}^{-1}\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{X}}^{-1}\underline{\underline{A}}\,\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{X}}^{-1}\underline{\underline{A}}^{2}\underline{\underline{X}}$$

Ex. 5) Diagonalization

Transformation of Forms to Principal Axes

Quadratic form: $Q = \underline{\underline{x}}^T \underline{\underline{A}} \underline{\underline{x}}$

If **A** is real symmetric, **A** has an orthogonal basis of n eigenvectors

→ **X** is orthogonal. $\underline{\underline{X}}^T = \underline{\underline{X}}^{-1}$ $\qquad \underline{\underline{A}} = \underline{\underline{X}}\underline{\underline{D}}\underline{\underline{X}}^{-1} = \underline{\underline{X}}\underline{\underline{D}}\underline{\underline{X}}^T$

$$Q = \underline{\underline{x}}^T \underline{\underline{X}}\underline{\underline{D}}\underline{\underline{X}}^T \underline{\underline{x}} = \underline{y}^T \underline{\underline{D}}\underline{y} = \lambda_1 y_1^2 + \lambda_1 y_n^2 + \cdots + \lambda_n y_n^2$$

$$\left(\underline{y} = \underline{\underline{X}}^T \underline{x} = \underline{\underline{X}}^{-1}\underline{x},\ \ \underline{x} = \underline{\underline{X}}\underline{y}\right)$$

Theorem 6: The substitution $\underline{x}$=**X**$\underline{y}$, transforms a quadratic form

$$Q = \underline{\underline{x}}^T \underline{\underline{A}} \underline{\underline{x}} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} x_j x_k$$

to the principal axes form $Q = \underline{y}^T \underline{\underline{D}}\underline{y} = \lambda_1 y_1^2 + \lambda_1 y_n^2 + \cdots + \lambda_n y_n^2$

where $\lambda_1, \ldots, \lambda_n$ eigenvalues of the symmetric Matrix **A**, and **X** is orthogonal matrix with corresponding eigenvectors as column vectors.

Ex. 6) Conic sections.

Example) Solution of linear 1st-order Eqn.:

$$\dot{\underline{y}} = \frac{d\underline{y}}{dt} = \underline{\underline{A}}\,\underline{y}$$

$$\text{Define}: \underline{y} = \underline{\underline{X}}\,\underline{z} \;\rightarrow\; \underline{z} = \underline{\underline{X}}^{-1}\underline{y}$$

$$\underline{\underline{X}}\dot{\underline{z}} = \underline{\underline{A}}\,\underline{\underline{X}}\,\underline{z} \;\rightarrow\; \dot{\underline{z}} = \underline{\underline{D}}\,\underline{z}$$

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix} \Rightarrow \begin{bmatrix}z_1(t)\\ z_2(t)\end{bmatrix} = \begin{pmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{pmatrix}\begin{bmatrix}z_1(0)\\ z_2(0)\end{bmatrix}$$

$$\dot{\underline{z}} = \underline{\underline{D}}\,\underline{z} \Rightarrow \dot{\underline{z}} = e^{\underline{\underline{D}}t}\underline{z}(0)$$

$$\Rightarrow \underline{y}(t) = \underline{\underline{X}}\,\underline{z}(t) = \underline{\underline{X}}e^{\underline{\underline{D}}t}\underline{\underline{X}}^{-1}\underline{y}(0)$$

Ex.) $\dot{y}_1 = -0.5y_1 + y_2$ $\quad \Rightarrow \underline{y}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ for $\lambda_1 = -0.5$, $\underline{y}_2 = \begin{bmatrix}-0.5547\\0.8321\end{bmatrix}$ for $\lambda_2 = -2$

$\dot{y}_2 = -2y_2$

$$\underline{y}(t) = \begin{pmatrix}1 & -0.5547\\ 0 & 0.8321\end{pmatrix}\begin{pmatrix}e^{-0.5t} & 0\\ 0 & e^{-2t}\end{pmatrix}\begin{pmatrix}1 & -0.5547\\ 0 & 0.8321\end{pmatrix}^{-1}\underline{y}(0)$$