11.5. Heat Equation: Solution by Fourier Series

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u \quad \left(c^2 = \frac{K}{\rho\sigma} \right)$$

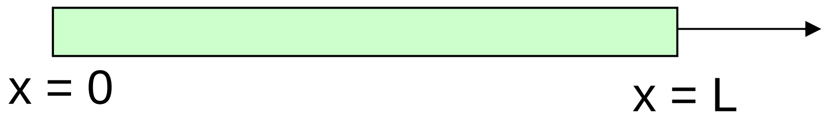
u: temperature, c^2: thermal diffusivity
ρ: density, σ: specific heat

u(t,x,y,z)

→ <u>Simple case</u>: temp. distribution in a long thin bar or wire: u(t,x)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Parabolic Eqn. (B^2-4AC=0)



B.C.s: u(0,t) = 0, u(L,t) = 0 for all t

I.C.: u(x,0) = f(x) (*f(0) = f(L) = 0*)

- *Same procedure as Section 11.3*

First Step: Two ODEs

let u(x,t) = F(x)G(t)

$$F'' + p^2 F = 0$$

$$\dot{G} + c^2 p^2 G = 0$$

$$u(0,t) = F(0)G(t) = 0$$
$$u(L,t) = F(L)G(t) = 0$$

Second Step: Satisfying the B.C.'s

$$F'' + p^2 F = 0 \;\Rightarrow\; F(x) = A\cos px + B\sin px \;\leftarrow\; \text{apply B.C's}: F(0) = F(L) = 0$$

$$\Rightarrow F_n(x) = \sin\frac{n\pi}{L}x \quad \text{for } B = 1 \;\; (n = 1, 2, \cdots)$$

(For n = 0 ? → F = 0 no interest)

$$\dot{G}+\lambda_n^2 G=0 \quad \left(\lambda_n=\frac{cn\pi}{L}\right) \Rightarrow G_n(t)=B_n e^{-\lambda_n^2 t} \quad (n=1,2,\cdots)$$

$$u_n(x,t)=B_n\left(\sin\frac{n\pi}{L}x\right)e^{-\lambda_n^2 t} \quad (n=1,2,\cdots)$$

Eigenfunctions with eigenvalues λ_n

Third Step: Solution of the Entire Problem

(all the terms → 0 as time increases)

- General solution: $$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}B_n\left(\sin\frac{n\pi}{L}x\right)e^{-\lambda_n^2 t} \quad \left(\lambda_n=\frac{cn\pi}{L}\right)$$

I.C.: $$u(x,0)=\sum_{n=1}^{\infty}B_n\left(\sin\frac{n\pi}{L}x\right)=f(x); \quad B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \quad (n=1,2,\cdots)$$

(Fourier sine series)

Ex.1~3)

Ex. 4) Bar with insulated ends.: B.C.'s: $u_x(0,t)$ = F'(0)G(t) = 0, $u_x(L,t)$ = F'(L)G(t) = 0

$$F''+p^2F=0 \Rightarrow F(x)=A\cos px+B\sin px \leftarrow \text{apply B.C's}: F'(0)=F'(L)=0$$

$$\Rightarrow F_n(x)=\cos\frac{n\pi}{L}x \quad \text{for } A=1 \ (n=0,2,\cdots)$$ *(n includes 0 !!!)*

$$u_n(x,t) = A_n\left(\cos\frac{n\pi x}{L}\right)e^{-\lambda_n^2 t} \quad (\lambda_n = \frac{cn\pi}{L}, n = 0,1,2,\cdots)$$

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} A_n\left(\cos\frac{n\pi}{L}x\right)e^{-\lambda_n^2 t} \quad \left(\lambda_n = \frac{cn\pi}{L}\right)$$

I.C.: $u(x,0) = \sum_{n=0}^{\infty} A_n\left(\cos\frac{n\pi}{L}x\right) = A_0 + \sum_{n=1}^{\infty} A_n\left(\cos\frac{n\pi}{L}x\right) = f(x)$; *(Fourier cosine series)*

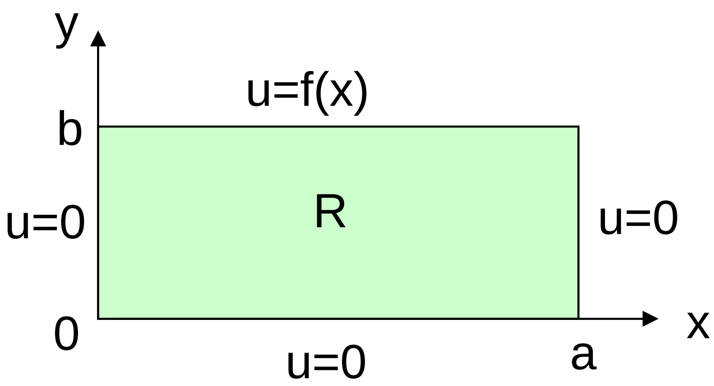
$$A_0 = \frac{1}{L}\int_0^L f(x)dx, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \quad (n = 1,2,\cdots)$$

Steady-State 2D Heat Flow

$$\frac{\partial u}{\partial t} = c^2\nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \Rightarrow \text{ at steady state}: \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \left(\because \frac{\partial u}{\partial t} = 0\right)$$

Elliptic Eqn. (B^2-4AC<0)

Dirichlet Problem in a Rectangle R



$$u(x,y)=F(x)G(y)\quad\Rightarrow \frac{1}{F}\frac{d^2F}{dx^2}=-\frac{1}{G}\frac{d^2G}{dy^2}=-k$$

$$\frac{d^2F}{dx^2}+kF=0 \leftarrow F(0)=0,\ F(a)=0 \quad \Rightarrow F=F_n(x)=\sin\frac{n\pi}{a}x\quad (n=1,2,\cdots)$$

$$\frac{d^2G}{dy^2}-\left(\frac{n\pi}{a}\right)^2 G=0 \quad\Rightarrow\quad G(y)=G_n(y)=A_n e^{n\pi y/a}+B_n e^{-n\pi y/a}$$

From $G_n(0) = 0$ at y=0: $G_n(y)=2A_n\sinh\frac{n\pi y}{a}=A_n^*\sinh\frac{n\pi y}{a}$

$$u_n(x,y)=A_n^*\sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a}$$

$$u(x,y)=\sum_{n=1}^{\infty}u_n(x,y)=\sum_{n=1}^{\infty}A_n^*\sinh\frac{n\pi y}{a}\sin\frac{n\pi x}{a}$$

B.C.: $u(x,b)=\sum_{n=1}^{\infty}\left(A_n^*\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a};$ $\quad A_n^*\sinh\frac{n\pi b}{a}=\frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$

11.6. Heat Equation: Solution by Fourier Integrals and Transforms

- In the case of infinite bars: Fourier series → Fourier integrals

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial t^2}$$ (laterally insulated, infinite ends), u(x,0) = f(x) (-∞ < x < ∞)

$$F'' + p^2 F = 0 \qquad \dot{G} + c^2 p^2 G = 0$$

$$\Rightarrow$$

$$F(x) = A\cos px + B\sin px; \quad G(t) = e^{-c^2 p^2 t}$$

$$u(x,t;p) = \left(A\cos px + B\sin px\right)e^{-c^2 p^2 t}$$

Use of Fourier Integrals

- f(x): nonperiodic function (p multiples of a fixed number) → use of Fourier integrals
  A & B: functions of p, then A(p) & B(p)

$$u(x,t) = \int_0^\infty u(x,t;p)dp = \int_0^\infty \left(A(p)\cos px + B(p)\sin px\right)e^{-c^2 p^2 t}dp$$

Determination of A(p) and B(p) from the Initial Condition

$$u(x,0) = \int_0^\infty \left(A(p)\cos px + B(p)\sin px\right)dp = f(x) \Rightarrow \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(v)\cos(px - pv)dv\right]dp$$

$$\left(A(p) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos pv dv, \quad B(p) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin pv dv\right)$$

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^{\infty} e^{-c^2p^2t}\cos(px - pv)dp\right]dv$$

$$\left(\int_0^{\infty} e^{-s^2}\cos 2bs\, ds = \frac{\sqrt{\pi}}{2}e^{-b^2}\right) \quad \left(\text{let } s^2 = c^2p^2t,\ b = \frac{x - v}{2c\sqrt{t}}\right)$$

$$\int_0^{\infty} e^{-c^2p^2t}\cos(px - pv)dp = \frac{\sqrt{\pi}}{2c\sqrt{t}}\exp\left\{-\frac{(x - v)^2}{4c^2t}\right\}$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\exp\left\{-\frac{(x - v)^2}{4c^2t}\right\}dv$$

$$\left(\text{let } z = (v - x)/(2c\sqrt{t})\right) \quad u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x + 2cz\sqrt{t})e^{-z^2}\, dz$$

Ex. 1)

Use of Fourier Transforms

Ex. 2) Temperature in the infinite bar

- Fourier transform w.r.t. x and resulting ODE in t

$$\Im(u_t) = c^2\Im(u_{xx}) = -c^2w^2\hat{u} \qquad (\text{let } \hat{u} = \Im(u))$$

$$\Im(u_t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t e^{-iwx}dx = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} ue^{-iwx}dx = \frac{\partial\hat{u}}{\partial t}$$

$$\frac{\partial\hat{u}}{\partial t} = -c^2w^2\hat{u}$$

$$\hat{u}(w,t) = C(w)e^{-c^2w^2t}$$

$$\text{I.C.}: \hat{u}(w,0) = C(w) = \hat{f}(w) \Rightarrow \hat{u}(w,t) = \hat{f}(w)e^{-c^2w^2t}$$

$$\text{Inversion}: u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{-c^2w^2t}e^{iwx}dw \qquad \left(\because \hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iwv}dv\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{-\infty}^{\infty} e^{-c^2w^2t}e^{i(wx-wv)}dw\right]dv$$

(Imaginary part: odd func)

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{-\infty}^{\infty} e^{-c^2w^2t}\cos(wx-wv)dw\right]dv$$

Ex. 3) Convolution method

$$\text{Starting from}: u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(w)e^{-c^2w^2t}e^{iwx}dw \qquad \hat{g}(w) = \frac{1}{\sqrt{2\pi}}e^{-c^2w^2t}$$

$$\left(\text{use } u(x,t) = f * g(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty}\hat{f}(w)\hat{g}(w)e^{iwv}dw\right)$$

$$\left(\Im(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-w^2/4a} \Rightarrow \Im(e^{-x^2/4c^2t}) = \sqrt{2c^2t}e^{-c^2w^2} = \sqrt{2c^2t}\sqrt{2\pi}\hat{g}(w) \right)$$

$$\left\{ \frac{1}{4a} = c^2 t \right\}$$

(See Ex. 2 of Sec. 10.10 for the Fourier transform)

$$\text{Inverse of } \hat{g}(w) \Rightarrow \frac{1}{\sqrt{2c^2t}\sqrt{2\pi}} e^{-x^2/4c^2t}$$

$$u(x,t) = (f * g)(x) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(p) \exp\left\{ -\frac{(x-p)^2}{4c^2 t} \right\} dp$$

Ex. 4) Fourier sine transform applied to the heat equation

A laterally insulated bar from x=0 to infinity, (u(x,0) = f(x), u(0,t) = 0)

$$\frac{\partial \hat{u}_s}{\partial t} = -c^2 w^2 \hat{u}_s \quad \leftarrow \Im_S(f''(x)) = -w^2 \Im_S(f(x)) + \sqrt{\frac{2}{\pi}} w f(0)$$

$$\Rightarrow \hat{u}_s(w,t) = \hat{f}_s(w) e^{-c^2 w^2 t} \qquad \left(\hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(p) \sin wp \, dp \right)$$

$$u(x,t) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(p) \sin wp \, e^{-c^2 w^2 t} \sin wx \, dp \, dw$$

Example) Heating a semi-infinite slab

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$ I.C.: u(x,0) = 0, B.C.'s: u(0,t) = 1, u(∞,t) = 0

Define a new variable: $\eta = \frac{x}{\sqrt{4\alpha t}}$

$$\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\eta}{t}\frac{du}{d\eta}; \quad \frac{\partial^2 u}{\partial x^2} = \frac{d^2 u}{d\eta^2}\frac{1}{4\alpha t}$$

$$\Rightarrow \frac{d^2 u}{d\eta^2} + 2\eta \frac{du}{d\eta} = 0 \text{ with B.C.s}: u = 1 \text{ at } \eta = 0,\ u = 0 \text{ at } \eta = \infty$$

$$\text{let } \psi = \frac{du}{d\eta}, \text{ then } \psi = C_1 \exp(-\eta^2),\ u = C_1 \int_0^{\eta} \exp(-\bar{\eta}^2) d\bar{\eta} + C_2$$

$$u(\eta) = 1 - \frac{\int_0^{\eta} \exp(-\bar{\eta}^2) d\bar{\eta}}{\int_0^{\infty} \exp(-\bar{\eta}^2) d\bar{\eta}} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} \exp(-\bar{\eta}^2) d\bar{\eta} = 1 - \operatorname{erf}(\eta)$$

error function

(See Ex. 2 of Sec. 10.10 for the integration of special function)