CHE302 LECTURE IX
FREQUENCY RESPONSES

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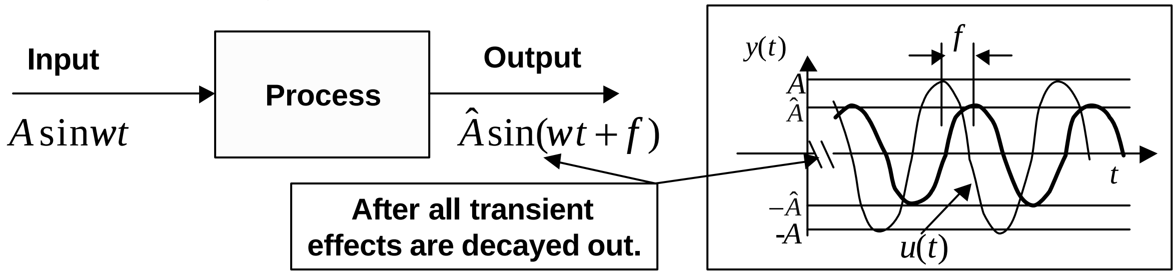
DEFINITION OF FREQUENCY RESPONSE

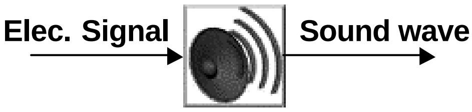
- **For linear system**
  - **"The ultimate output response of a process for a sinusoidal input at a frequency will show amplitude change and phase shift at the same frequency depending on the process characteristics."**

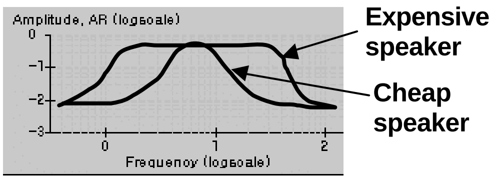


  - **Amplitude ratio (AR): attenuation of amplitude,** $\hat{A}/A$
  - **Phase angle (** f **): phase shift compared to input**
  - **These two quantities are the function of frequency.**

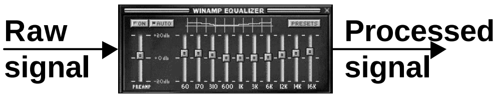
BENEFITS OF FREQUENCY RESPONSE

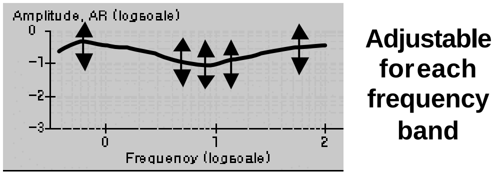
- **Frequency responses are the informative representations of dynamic systems**
  - **Audio Speaker**



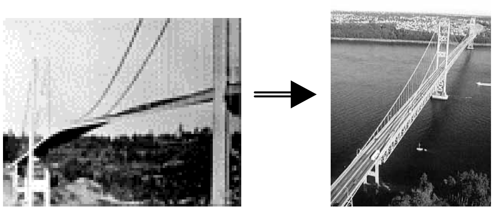


  - **Equalizer**

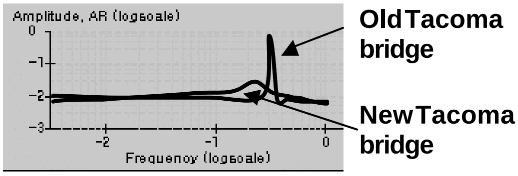




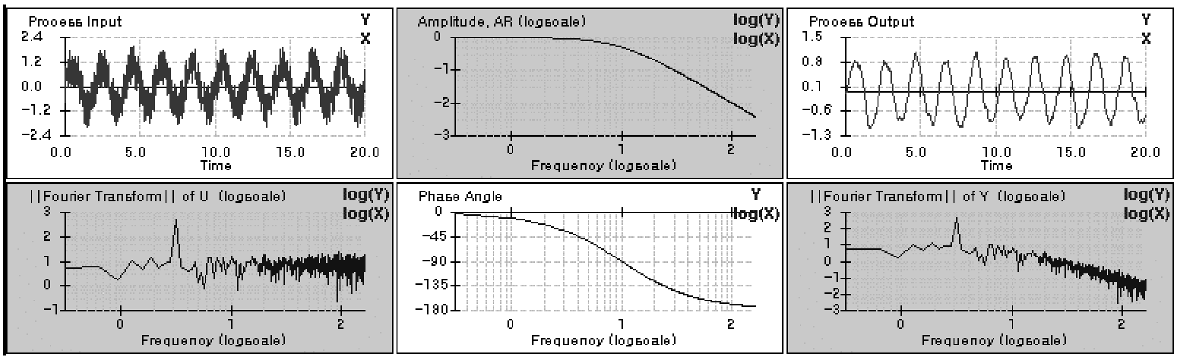
  - **Structure**



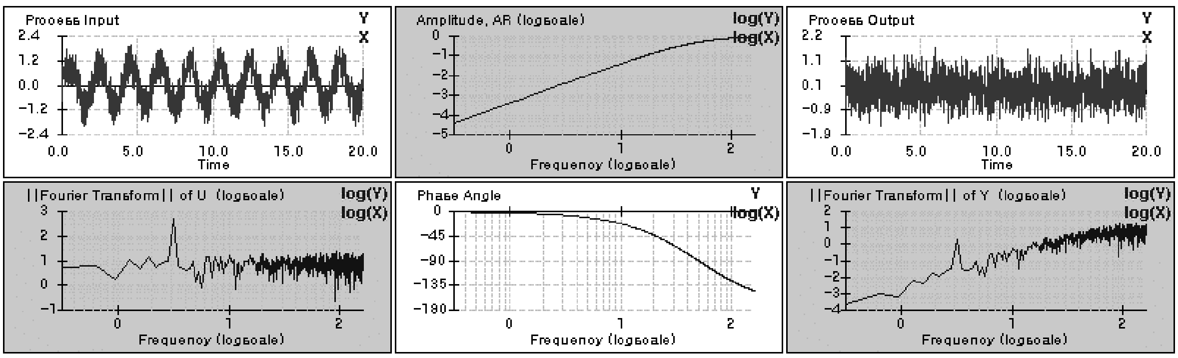
Old Tacoma bridge   New Tacoma bridge



  - **Low-pass filter**



  - **High-pass filter**

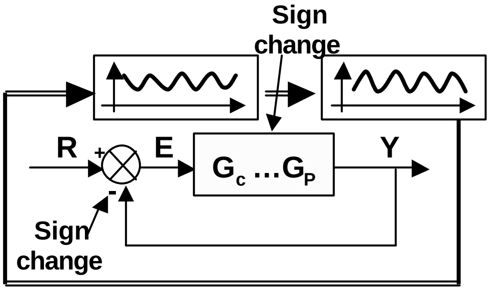


  - **In signal processing field, transfer functions are called "filters".**

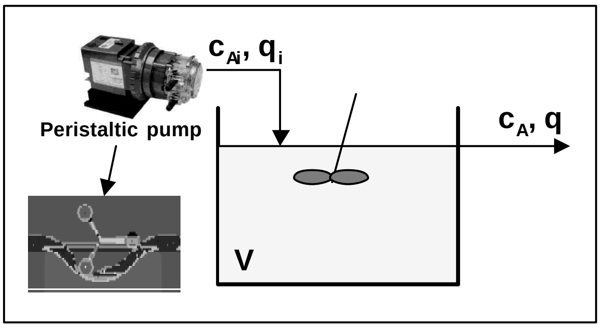
- **Any linear dynamical system is completely defined by its frequency response.**
  - The AR and phase angle define the system completely.
  - Bode diagram
    - AR in log-log plot
    - Phase angle in log-linear plot
  - Via efficient numerical technique (fast Fourier transform, FFT), the output can be calculated for any type of input.

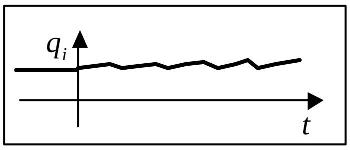
- **Frequency response representation of a system dynamics is very convenient for designing a feedback controller and analyzing a closed-loop system.**
  - Bode stability
  - Gain margin (GM) and phase margin (PM)

- **Critical frequency**
  - As controller gain changes, the amplitude ratio (AR) and the phase angle (PA) change.
  - The frequency where the PA reaches –180° is called critical frequency (w_c).
  - The component of output at the critical frequency will have the exactly same phase as the signal goes through the loop due to comparator (-180 °) and phase shift of the process (-180 °).
  - For the open-loop gain at the critical frequency, $K_{OL}(w_c) = 1$
    - No change in magnitude
    - Continuous cycling
  - For $K_{OL}(w_c) > 1$
    - Getting bigger in magnitude
    - Unstable
  - For $K_{OL}(w_c) < 1$
    - Getting smaller in magnitude
    - Stable



- **Example**
  - **If a feed is pumped by a peristaltic pump to a CSTR, will the fluctuation of the feed flow appear in the output?**



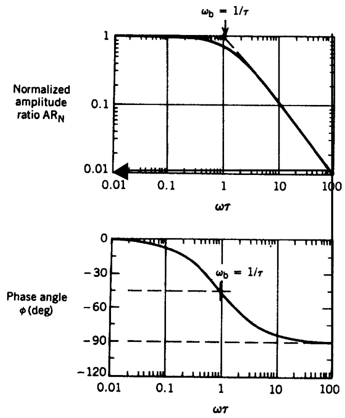


$$V\frac{dc_A}{dt} = q_i c_{Ai} - q c_A \ (q \approx \text{constant})$$

$$\frac{C_A(s)}{q_i(s)} = \frac{C_{Ai}}{Vs+q} = \frac{C_{Ai}/q}{(V/q)s+1}$$

  - **V=50cm^3, q=90cm^3/min (so is the average of q_i)**
    - **Process time constant=0.555min.**
  - **The rpm of the peristaltic pump is 60rpm.**
    - **Input frequency=180rad/min (3blades)**
  - **The AR=0.01** $(wt = 100)$

If the magnitude of fluctuation of q_i is 5% of nominal flow rate, the fluctuation in the output concentration will be about 0.05% which is almost unnoticeable.



OBTAINING FREQUENCY RESPONSE

- **From the transfer function, replace** s **with** jw

$$G(s) \xrightarrow{s=jw} G(jw)$$

Transfer function **Frequency response**

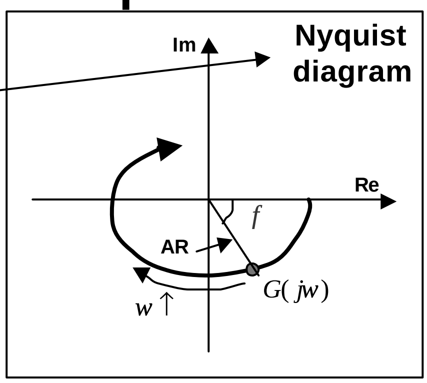
  - **For a pole,** $s = a + jw$ **, the response mode is** $e^{(a+jw)t}$ **.**
  - **If the modes are not unstable (**$a \le 0$**) and enough time elapses, the survived modes becomes** e^{jwt}**. (ultimate response)**
- **The frequency response,** $G(jw)$ **is complex as a function of frequency.**

$$G(jw) = \text{Re}[G(jw)] + j\,\text{Im}[G(jw)]$$

$$AR = |G(jw)| = \sqrt{\text{Re}[G(jw)]^2 + \text{Im}[G(jw)]^2}$$

$$f = \measuredangle G(jw) = \tan^{-1}\left(\text{Im}[G(jw)]/\text{Re}[G(jw)]\right)$$

Bode plot



- **Getting ultimate response**
  - **For a sinusoidal forcing function** $Y(s) = G(s)\dfrac{Aw}{s^2 + w^2}$
  - **Assume** $G(s)$ **has stable poles** b_i**.**

Decayed out at large t

$$Y(s) = G(s)\frac{Aw}{s^2 + w^2} = \frac{a_1}{s + b_1} + \cdots + \frac{a_n}{s + b_n} + \frac{Cs + Dw}{s^2 + w^2}$$

$$G(jw)Aw = Cjw + Dw \Rightarrow G(jw) = \frac{D}{A} + j\frac{C}{A} = R + jI$$

$$C = IA,\ D = RA \ \Rightarrow\ y_{ul} = A(I\cos wt + R\sin wt) = \hat{A}\sin(wt + f)$$

$$\therefore AR = \hat{A}/A = \sqrt{R^2 + I^2} = \left|G(jw)\right| \text{ and } f = \tan^{-1}(I/R) = \measuredangle G(jw)$$

  - **Without calculating transient response, the frequency response can be obtained directly from** $G(jw)$ **.**
  - **Unstable transfer function does not have a frequency response because a sinusoidal input produces an unstable output response.**

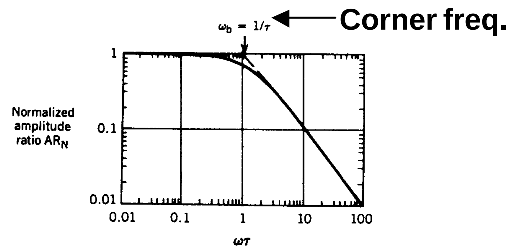
- **First-order process**

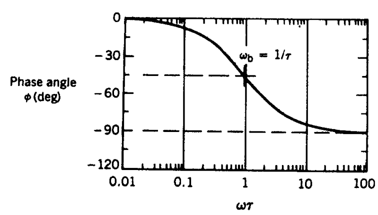
$$G(s) = \frac{K}{(ts + 1)}$$

$$G(jw) = \frac{K}{(1 + jwt)} = \frac{K}{(1 + w^2 t^2)}(1 - jwt)$$

$$AR_N = \left|G(jw)\right| = \frac{1}{\sqrt{1 + w^2 t^2}}$$

$$f = \measuredangle G(jw) = -\tan^{-1}(wt)$$





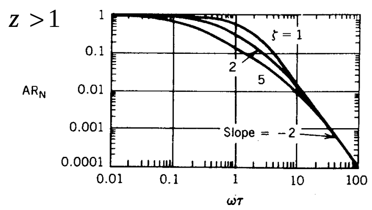
- **Second-order process**

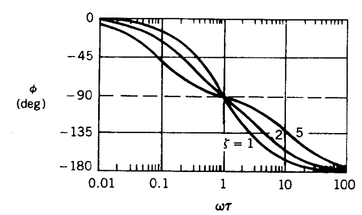
$$G(s) = \frac{K}{(t^2 s^2 + 2z t s + 1)}$$

$$G(jw) = \frac{K}{(1 - t^2 w^2) + 2jz t w}$$

$$AR = \left|G(jw)\right| = \frac{K}{\sqrt{(1 - w^2 t^2)^2 + (2z w t)^2}}$$

$$f = \measuredangle G(jw) = \tan^{-1}\frac{\text{Im}(G(jw))}{\text{Re}(G(jw))} = -\tan^{-1}\frac{2zwt}{1 - w^2 t^2}$$





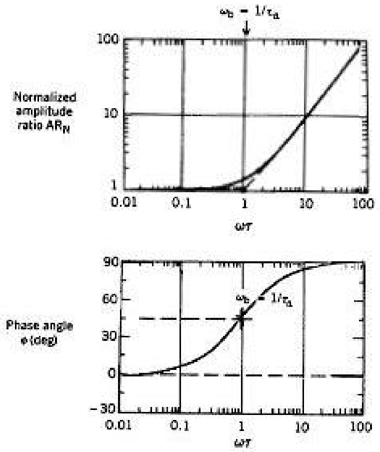
- **Process Zero (lead)**

$$G(s)=t_a s+1$$

$$G(jw)=1+jwt_a$$

$$AR_N=\left|G(jw)\right|=\sqrt{1+w^2t_a^2}$$

$$f=\measuredangle G(jw)=\tan^{-1}(wt_a)$$



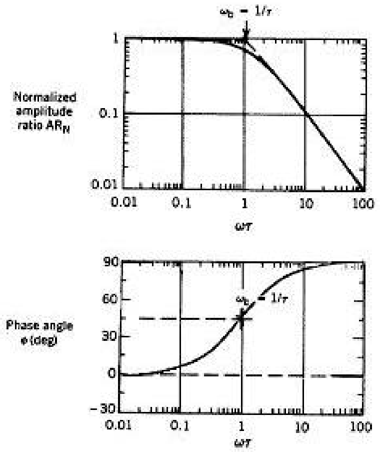
- **Unstable pole**

$$G(s)=\frac{1}{(-ts+1)}$$

$$G(jw)=\frac{1}{1-jtw}=\frac{1}{1+t^2w^2}(1+jtw)$$

$$AR=\left|G(jw)\right|=\frac{1}{\sqrt{1+w^2t^2}}$$

$$f=\measuredangle G(jw)=\tan^{-1}\frac{\mathrm{Im}(G(jw))}{\mathrm{Re}(G(jw))}=\tan^{-1}wt$$

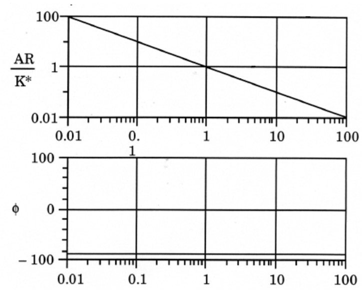


- **Integrating process**

$$G(s)=\frac{1}{As}\qquad G(jw)=\frac{1}{jAw}=-\frac{1}{Aw}j$$

$$AR_N=\left|G(jw)\right|=\frac{1}{Aw}$$

$$f=\measuredangle G(jw)=\tan^{-1}(-\frac{1}{0\cdot w})=-\frac{p}{2}$$

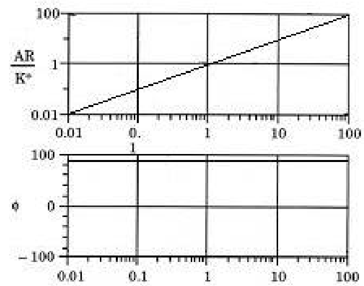


- **Differentiator**

$$G(s)=As\qquad G(jw)=jAw$$

$$AR_N=\left|G(jw)\right|=Aw$$

$$f=\measuredangle G(jw)=\tan^{-1}(\frac{1}{0\cdot w})=\frac{p}{2}$$



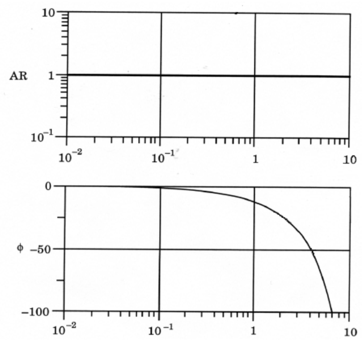
- **Pure delay process**

$$G(s)=e^{-qs}$$

$$G(jw)=e^{-jqw}=\cos qw-j\sin qw$$

$$AR=\left|G(jw)\right|=1$$

$$f=\measuredangle G(jw)=-\tan^{-1}\tan qw=-qw$$



SKETCHING BODE PLOT

$$G(s)=\frac{G_a(s)G_b(s)G_c(s)\cdots}{G_1(s)G_2(s)G_3(s)\cdots} \qquad G(jw)=\frac{G_a(jw)G_b(jw)G_c(jw)\cdots}{G_1(jw)G_2(jw)G_3(jw)\cdots}$$

$$|G(jw)|=\frac{|G_a(jw)||G_b(jw)||G_c(jw)|\cdots}{|G_1(jw)||G_2(jw)||G_3(jw)|\cdots}$$

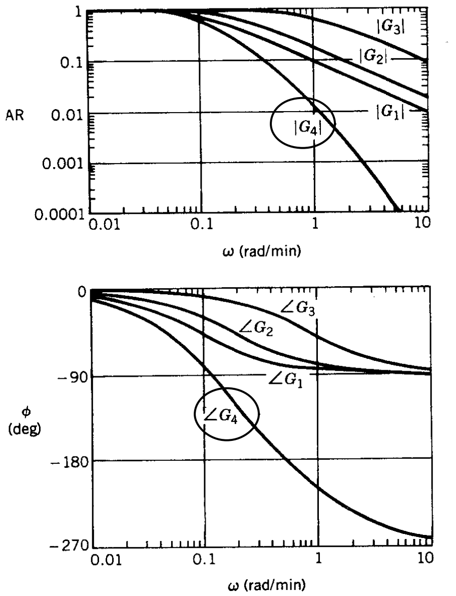
$$\measuredangle G(jw)=\measuredangle G_a(jw)+\measuredangle G_b(jw)+\measuredangle G_c(jw)+\cdots$$
$$-\measuredangle G_1(jw)-\measuredangle G_2(jw)-\measuredangle G_3(jw)-\cdots$$

- **Bode diagram**
  - AR vs. frequency in log-log plot
  - PA vs. frequency in semi-log plot
  - Useful for
    - Analysis of the response characteristics
    - Stability of the closed-loop system only for open-loop stable systems with phase angle curves exhibit a single critical frequency.

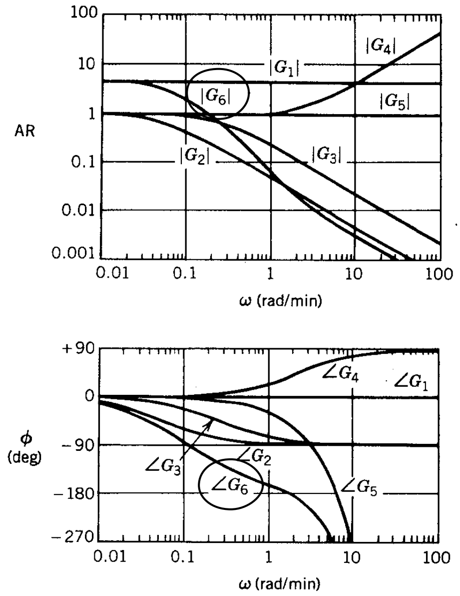
- **Amplitude Ratio on log-log plot**
  - Start from steady-state gain at w =0. If G_{OL} includes either integrator or differentiator it starts at ∞ or 0.
  - Each first-order lag (lead) adds to the slope –1 (+1) starting at the corner frequency.
  - Each integrator (differentiator) adds to the slope –1 (+1) starting at zero frequency.
  - A delays does not contribute to the AR plot.
- **Phase angle on semi-log plot**
  - Start from 0° or -180° at w =0 depending on the sign of steady-state gain.
  - Each first-order lag (lead) adds 0° to phase angle at w =0, adds -90° (+90°) to phase angle at $w=\infty$, and adds -45° (+45°) to phase angle at corner frequency.
  - Each integrator (differentiator) adds -90° (+90°) to the phase angle for all frequency.
  - A delay adds $-\theta w$ to phase angle depending on the frequency.

Examples

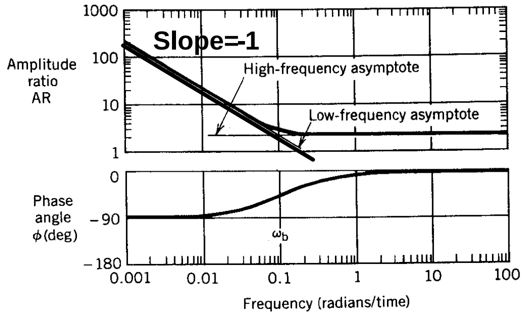
1. $$G(s)=\frac{K}{\underbrace{(10s+1)}_{G_1}\underbrace{(5s+1)}_{G_2}\underbrace{(s+1)}_{G_3}}$$



2. $$G(s)=\frac{\overset{G_1}{5}\,\overset{G_5}{(0.5s+1)}\,\overset{G_4}{e^{-0.5s}}}{\underbrace{(20s+1)}_{G_2}\underbrace{(4s+1)}_{G_3}}$$

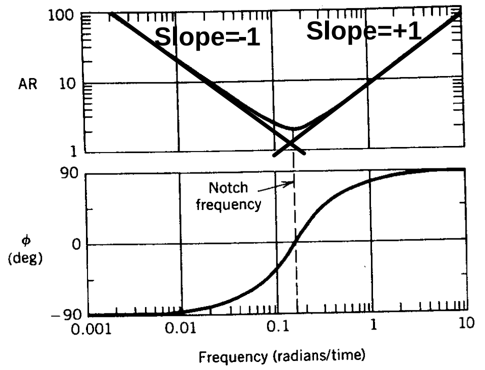


3. PI: $$G(s)=K_C\left(1+\frac{1}{\tau_I s}\right)$$



$$\omega_b = 1/\tau_I \text{ at } \phi = -45^\circ$$

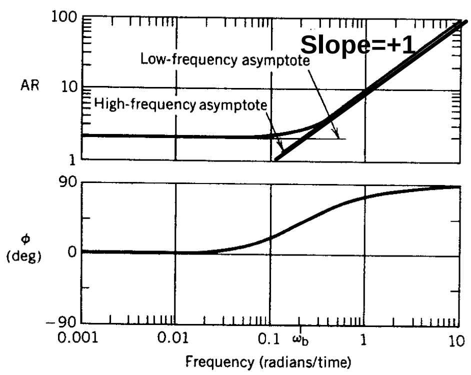
5. PID: $$G(s)=K_C\left(1+\frac{1}{\tau_I s}+\tau_D s\right)$$



$$\omega_{Notch} = 1/\sqrt{\tau_I \tau_D} \text{ at } \phi = 0^\circ$$

4. PD: $$G(s)=K_C\left(1+\tau_D s\right)$$

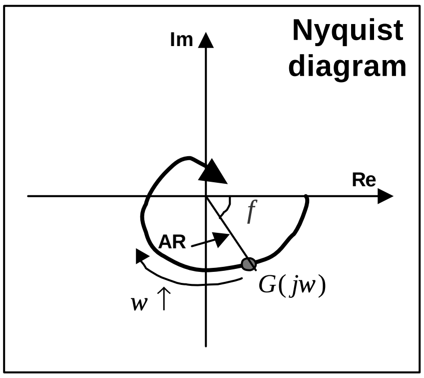
$$\omega_b = 1/\tau_D \text{ at } \phi = -45^\circ$$



NYQUIST DIAGRAM

- **Alternative representation of frequency response**
- **Polar plot of $G(jw)$ (w is implicit)**

$$G(jw) = \text{Re}[G(jw)] + j\,\text{Im}[G(jw)]$$



- **Compact (one plot)**
- **Wider applicability of stability analysis than Bode plot**
- **High frequency characteristics will be shrunk near the origin.**
  - **Inverse Nyquist diagram: polar plot of** $1/G(jw)$
- **Combination of different transfer function components is not easy as with Nyquist diagram as with Bode plot.**