CHE302 LECTURE V
LAPLACE TRANSFORM AND TRANSFER FUNCTION

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SOLUTION OF LINEAR ODE

- **1st-order linear ODE**
  - **Integrating factor:** For $\dfrac{dx}{dt} + a(t)x = f(x)$, I.F. $= \exp(\int a(t)dt)$

$$[xe^{\int a(t)dt}]' = f(t)e^{\int a(t)dt} \longrightarrow x(t) = [\int f(t)e^{\int a(t)dt}dt + C]e^{-\int a(t)dt}$$

- **High-order linear ODE with constant coeffs.**
  - **Modes: roots of characteristic equation**

For $a_2 x'' + a_1 x' + a_0 x = f(t)$,

$$a_2 p^2 + a_1 p + a_0 = a_2(p - p_1)(p - p_1) = 0$$

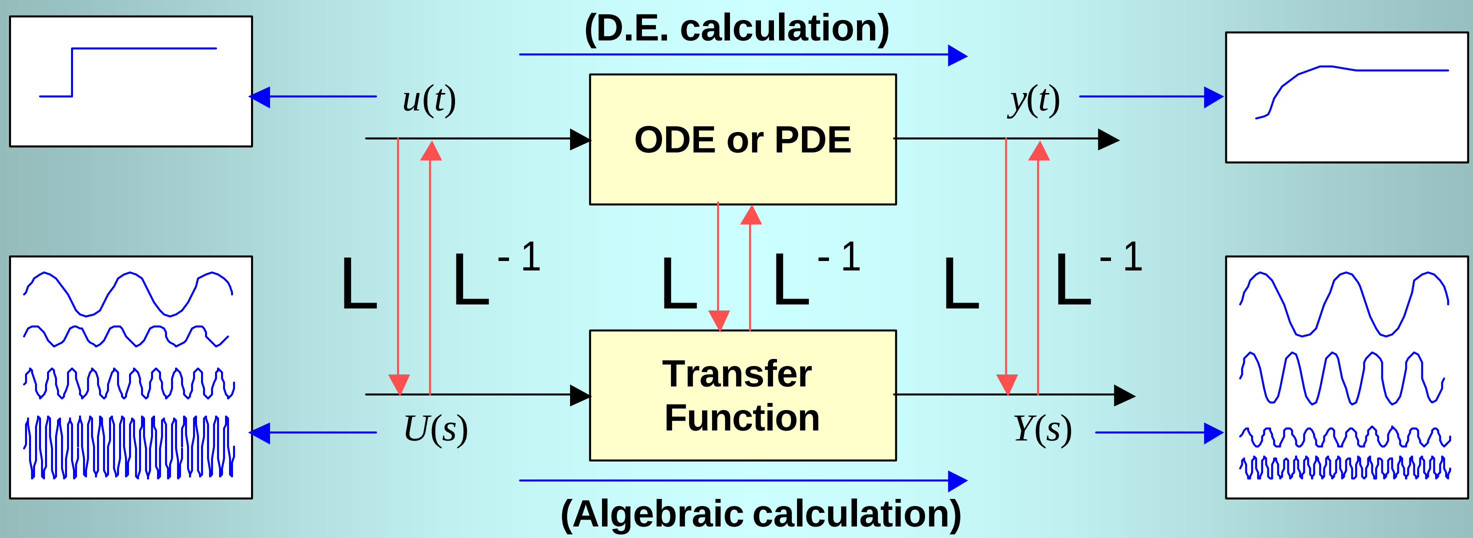
  - **Depending on the roots, modes are**
    - **Distinct roots:** $(e^{-p_1 t}, e^{-p_2 t})$
    - **Double roots:** $(e^{-p_1 t}, te^{-p_1 t})$
    - **Imaginary roots:** $(e^{-at}\cos bt, e^{-at}\sin bt)$

$\longrightarrow$ **Solution is a linear combination of modes and the coefficients are decided by the initial conditions.**

- **Many other techniques for different cases**

LAPLACE TRANSFORM FOR LINEAR ODE AND PDE

- **Laplace Transform**
  - Not in time domain, rather in frequency domain
  - Derivatives and integral become some operators.
  - ODE is converted into algebraic equation
  - PDE is converted into ODE in spatial coordinate
  - Need inverse transform to recover time-domain solution



DEFINITION OF LAPLACE TRANSFORM

- **Definition**

$$F(s) = \mathcal{L}\{f(t)\} \triangleq \int_0^\infty f(t)e^{-st}dt$$

  - *F(s)* **is called** ***Laplace transform*** **of** *f(t)*.
  - *f(t)* **must be piecewise continuous.**
  - *F(s)* **contains no information on** *f(t)* **for** $t < 0$.
  - **The past information on** *f(t)* **(for** $t < 0$**) is irrelevant.**
  - **The** *s* **is a complex variable called "*Laplace transform variable*"**

- **Inverse Laplace transform**

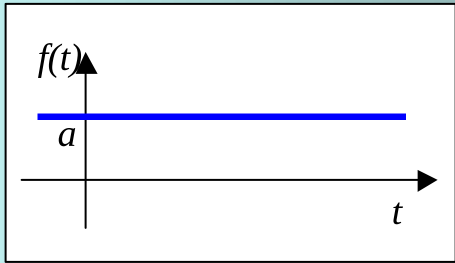
$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

  - $\mathcal{L}$ **and** $\mathcal{L}^{-1}$ **are linear.** $\mathcal{L}\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$

LAPLACE TRANSFORM OF FUNCTIONS

- **Constant function,** a

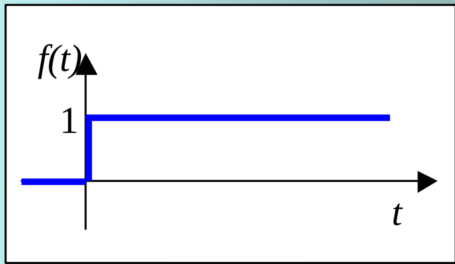
$$\mathcal{L}\{a\} = \int_0^{\infty} a e^{-st} dt = -\frac{a}{s} e^{-st} \Big|_0^{\infty} = 0 - \left(-\frac{a}{s}\right) = \frac{a}{s}$$



- **Step function, S(**t**)**

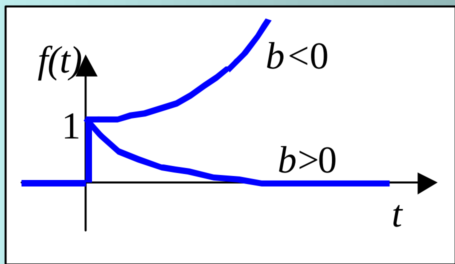
$$f(t) = \mathrm{S}(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\mathcal{L}\{\mathrm{S}(t)\} = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

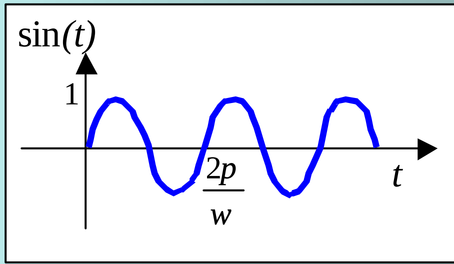


- **Exponential function,** e^{-bt}

$$\mathcal{L}\{e^{-bt}\} = \int_0^{\infty} e^{-bt} e^{-st} dt = \frac{-1}{s+b} e^{-(b+s)t} \Big|_0^{\infty} = \frac{1}{s+b}$$



- **Trigonometric functions**
  - **Euler's Identity:** $e^{jwt} \triangleq \cos wt + j\sin wt$



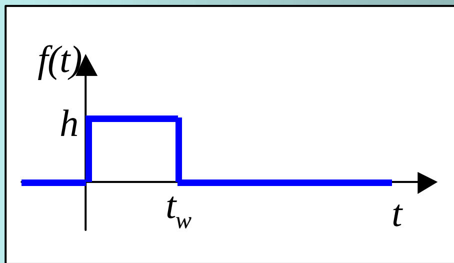
$$\cos wt = \frac{1}{2}\left(e^{jwt} + e^{-jwt}\right) \qquad \sin wt = \frac{1}{2j}\left(e^{jwt} - e^{-jwt}\right)$$

$$\mathrm{L}\{\sin wt\} = \mathrm{L}\left\{\frac{1}{2j}e^{jwt}\right\} - \mathrm{L}\left\{\frac{1}{2j}e^{-jwt}\right\} = \frac{1}{2j}\left(\frac{1}{s-jw} - \frac{1}{s+jw}\right) = \frac{w}{s^2+w^2}$$

$$\mathrm{L}\{\cos wt\} = \mathrm{L}\left\{\frac{1}{2}e^{jwt}\right\} + \mathrm{L}\left\{\frac{1}{2}e^{-jwt}\right\} = \frac{1}{2}\left(\frac{1}{s-jw} + \frac{1}{s+jw}\right) = \frac{s}{s^2+w^2}$$

- **Rectangular pulse, P(t)**

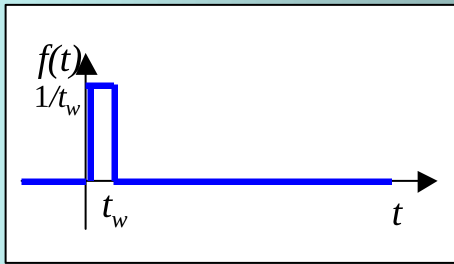
$$f(t) = \mathrm{P}(t) = \begin{cases} 0 & \text{for } t > t_w \\ h & \text{for } t_w \geq t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



$$\mathrm{L}\{\mathrm{P}(t)\} = \int_0^{t_w} h e^{-st} dt = -\frac{h}{s}e^{-st}\Big|_0^{t_w} = \frac{h}{s}\left(1 - e^{-t_w s}\right)$$

- **Impulse function,** $d(t)$

$$f(t) = d(t) = \lim_{t_w \to 0} \begin{cases} 0 & \text{for } t > t_w \\ 1/t_w & \text{for } t_w \geq t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



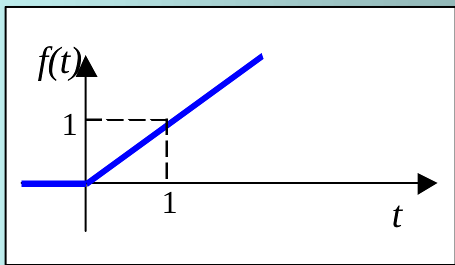
$$\mathcal{L}\{d(t)\} = \lim_{t_w \to 0} \int_0^{t_w} \frac{1}{t_w} e^{-st} dt = \lim_{t_w \to 0} \frac{1}{t_w s}\left(1 - e^{-t_w s}\right) = 1$$

$$\left(\text{L'Hospital's rule: } \lim_{t \to 0} \frac{f(t)}{g(t)} = \lim_{t \to 0} \frac{f'(t)}{g'(t)} \right)$$

- **Ramp function,** t

$$\mathcal{L}\{t\} = \int_0^{\infty} t e^{-st} dt$$

$$= \frac{t}{-s} e^{-st} \Big|_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} dt = \frac{1}{s} \int_0^{\infty} e^{-st} dt = \frac{1}{s^2}$$



$$\left(\text{Integration by part: } \int_0^{\infty} f' \cdot g dt = f \cdot g \Big|_0^{\infty} - \int_0^{\infty} f \cdot g' dt \right)$$

- **Refer the Table 3.1 (Seborg et al.) for other functions**

Table 3.1 Laplace Transforms for Various Time-Domain Functions[a]

	$f(t)$	$F(s)$
1.	$\delta(t)$ (unit impulse)	1
2.	$S(t)$ (unit step)	$\frac{1}{s}$
3.	t (ramp)	$\frac{1}{s^2}$
4.	t^{n-1}	$\frac{(n-1)!}{s^n}$
5.	e^{-bt}	$\frac{1}{s+b}$
6.	$\frac{1}{\tau} e^{-t/\tau}$	$\frac{1}{\tau s + 1}$
7.	$\frac{t^{n-1}e^{-bt}}{(n-1)!}$ $(n > 0)$	$\frac{1}{(s+b)^n}$
8.	$\frac{1}{\tau^n (n-1)!} t^{n-1} e^{-t/\tau}$	$\frac{1}{(\tau s + 1)^n}$
9.	$\frac{1}{b_1 - b_2}(e^{-b_2 t} - e^{-b_1 t})$	$\frac{1}{(s+b_1)(s+b_2)}$
10.	$\frac{1}{\tau_1 - \tau_2}(e^{-t/\tau_1} - e^{-t/\tau_2})$	$\frac{1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
11.	$\frac{b_3 - b_1}{b_2 - b_1} e^{-b_1 t} + \frac{b_3 - b_2}{b_1 - b_2} e^{-b_2 t}$	$\frac{s + b_3}{(s+b_1)(s+b_2)}$
12.	$\frac{1}{\tau_1}\frac{\tau_1 - \tau_3}{\tau_1 - \tau_2} e^{-t/\tau_1} + \frac{1}{\tau_2}\frac{\tau_2 - \tau_3}{\tau_2 - \tau_1} e^{-t/\tau_2}$	$\frac{\tau_3 s + 1}{(\tau_1 s + 1)(\tau_2 s + 1)}$
13.	$1 - e^{-t/\tau}$	$\frac{1}{s(\tau s + 1)}$
14.	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$

Table 3.1 (*Continued*)

	$f(t)$	$F(s)$
15.	$\cos \omega t$	$\dfrac{s}{s^2 + \omega^2}$
16.	$\sin(\omega t + \phi)$	$\dfrac{\omega \cos \phi + s \sin \phi}{s^2 + \omega^2}$
17.	$e^{-bt} \sin \omega t$ (b, ω real)	$\dfrac{\omega}{(s + b)^2 + \omega^2}$
18.	$e^{-bt} \cos \omega t$ (b, ω real)	$\dfrac{s + b}{(s + b)^2 + \omega^2}$
19.	$\dfrac{1}{\tau\sqrt{1 - \zeta^2}} e^{-\zeta t/\tau} \sin(\sqrt{1 - \zeta^2}\, t/\tau)$ $(0 \le \lvert\zeta\rvert < 1)$	$\dfrac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$
20.	$1 + \dfrac{1}{\tau_2 - \tau_1}(\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2})$ $(\tau_1 \neq \tau_2)$	$\dfrac{1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
21.	$1 - \dfrac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta t/\tau} \sin[\sqrt{1 - \zeta^2}\, t/\tau + \psi]$ $\psi = \tan^{-1} \dfrac{\sqrt{1 - \zeta^2}}{\zeta}$ $(0 \le \lvert\zeta\rvert < 1)$	$\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
22.	$1 - e^{-\zeta t/\tau} [\cos(\sqrt{1 - \zeta^2}\, t/\tau)$ $+ \dfrac{\zeta}{\sqrt{1 - \zeta^2}} \sin(\sqrt{1 - \zeta^2}\, t/\tau)]$ $(0 \le \lvert\zeta\rvert < 1)$	$\dfrac{1}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}$
23.	$1 + \dfrac{\tau_3 - \tau_1}{\tau_1 - \tau_2} e^{-t/\tau_1} + \dfrac{\tau_3 - \tau_2}{\tau_2 - \tau_1} e^{-t/\tau_2}$ $(\tau_1 \neq \tau_2)$	$\dfrac{\tau_3 s + 1}{s(\tau_1 s + 1)(\tau_2 s + 1)}$
24.	$\dfrac{df}{dt}$	$sF(s) - f(0)$
25.	$\dfrac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

[a]Note that $f(t)$ and $F(s)$ are defined for $t \geq 0$ only.

PROPERTIES OF LAPLACE TRANSFORM

- **Differentiation**

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_0^\infty f' \cdot e^{-st} dt = f(t) e^{-st}\Big|_0^\infty - \int_0^\infty f \cdot (-s) e^{-st} dt \qquad \text{(by } i.b.p.\text{)}$$

$$= s\int_0^\infty f \cdot e^{-st} dt - f(0) = sF(s) - f(0)$$

$$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = \int_0^\infty f'' \cdot e^{-st} dt = f(t)' e^{-st}\Big|_0^\infty - \int_0^\infty f' \cdot (-s) e^{-st} dt = s\int_0^\infty f' \cdot e^{-st} dt - f'(0)$$

$$= s\left(sF(s) - f(0)\right) - f'(0) = s^2 F(s) - sf(0) - f'(0)$$

$$\vdots$$

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = \int_0^\infty f^{(n)} \cdot e^{-st} dt = f(t)^{(n-1)} e^{-st}\Big|_0^\infty - \int_0^\infty f^{(n-1)} \cdot (-s) e^{-st} dt$$

$$= s\int_0^\infty f^{(n-1)} \cdot e^{-st} dt - f^{(n-1)}(0) = s\left(\mathcal{L}\left\{\frac{d^{n-1} f}{dt^{n-1}}\right\}\right) - f^{(n-1)}(0)$$

$$= s^n F(s) - s^{n-1} f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

- **If** $f(0) = f'(0) = f''(0) = \ldots = f^{(n\text{-}1)}(0) = 0$**,**
  - **Initial condition effects are vanished.**
  - **It is very convenient to use deviation variables so that all the effects of initial condition vanish.**

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s)$$

$$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s^2 F(s)$$

$$\vdots$$

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s)$$

- **Transforms of linear differential equations.**

$$y(t) \xrightarrow{\mathcal{L}} Y(s), \quad u(t) \xrightarrow{\mathcal{L}} U(s)$$

$$\frac{dy(t)}{dt} \xrightarrow{\mathcal{L}} sY(s) \quad (\text{if } \; y(0) = 0)$$

$$t\,\frac{dy(t)}{dt} = -y(t) + Ku(t) \quad (y(0) = 0) \xrightarrow{\mathcal{L}} (t\,s + 1)Y(s) = KU(s)$$

$$\frac{\partial T_L}{\partial t} = -v\frac{\partial T_L}{\partial z} + \frac{1}{t_{HL}}(T_w - T_L) \xrightarrow{\mathcal{L}} t_{HL} v \frac{\partial \tilde{T}_L(s)}{\partial z} + (t_{HL} s + 1)\tilde{T}_L(s) = \tilde{T}_w(s)$$

- **Integration**

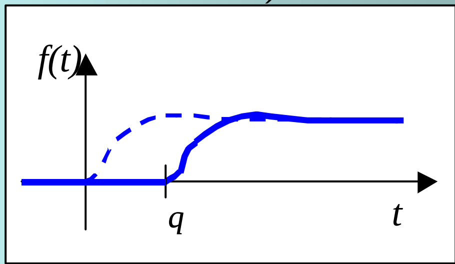
$$\mathrm{L}\left\{\int_0^t f(x)dx\right\} = \int_0^\infty \left(\int_0^t f(x)dx\right)e^{-st}dt$$

$$= \frac{e^{-st}}{-s}\int_0^t f(x)dx\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty f\cdot e^{-st}dt = \frac{F(s)}{s} \qquad \text{(by } i.b.p.\text{)}$$

(The boundary term is struck through with an arrow labelled 0.)

$$\left(\text{Leibniz rule: } \frac{d}{dt}\int_{a(t)}^{b(t)} f(t)dt = f(b(t))\frac{db(t)}{dt} - f(a(t))\frac{da(t)}{dt}\right)$$

- **Time delay (Translation in time)**



$$f(t) \xrightarrow{+q \text{ in } t} f(t-q)S(t-q)$$

$$\mathrm{L}\{f(t-q)S(t-q)\} = \int_q^\infty f(t-q)e^{-st}dt = \int_0^\infty f(t)e^{-s(t+q)}dt \qquad (\text{let } t = t-q)$$

$$= e^{-qs}\int_0^\infty f(t)e^{-ts}dt = e^{-qs}F(s)$$

- **Derivative of Laplace transform**

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty f\cdot e^{-st}dt = \int_0^\infty f\cdot\frac{d}{ds}e^{-st}dt = \int_0^\infty (-t\cdot f)e^{-st}dt = \mathrm{L}[-t\cdot f(t)]$$

- **Final value theorem**
  - **From the LT of differentiation, as s approaches to zero**

$$\lim_{s \to 0} \int_0^\infty \frac{df}{dt} \cdot e^{-st} dt = \lim_{s \to 0} \left[sF(s) - f(0) \right]$$

$$\int_0^\infty \frac{df}{dt} dt = f(\infty) - f(0) = \lim_{s \to 0} sF(s) - f(0) \Rightarrow f(\infty) = \lim_{s \to 0} sF(s)$$

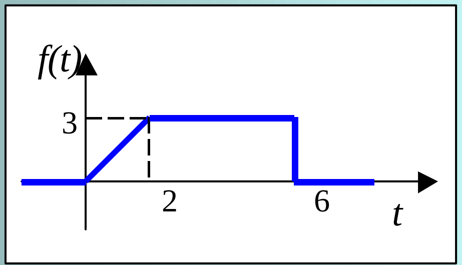
  - **Limitation: $f(\infty)$ has to exist. If it diverges or oscillates, this theorem is not valid.**

- **Initial value theorem**
  - **From the LT of differentiation, as s approaches to infinity**

$$\lim_{s \to \infty} \int_0^\infty \frac{df}{dt} \cdot e^{-st} dt = \lim_{s \to \infty} \left[sF(s) - f(0) \right]$$

$$\lim_{s \to \infty} \int_0^\infty \frac{df}{dt} e^{-st} dt = 0 = \lim_{s \to \infty} sF(s) - f(0) \Rightarrow f(0) = \lim_{s \to \infty} sF(s)$$

EXAMPLE ON LAPLACE TRANSFORM (1)

- 

$$f(t)=\begin{cases}1.5t & \text{for } 0\le t<2\\ 3 & \text{for } 2\le t<6\\ 0 & \text{for } 6\le t\\ 0 & \text{for } t<0\end{cases}$$

$$f(t)=1.5t\,S(t)-1.5(t-2)S(t-2)-3S(t-6)$$

$$\therefore F(s)=\mathcal{L}\{f(t)\}=\frac{1.5}{s^2}(1-e^{-2s})-\frac{3}{s}e^{-6s}$$

- For $F(s)=\dfrac{2}{s-5}$, find $f(0)$ and $f(\infty)$.
  - **Using the initial and final value theorems**

    $$f(0)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty}\frac{2s}{s-5}=2 \qquad f(\infty)=\lim_{s\to 0} sF(s)=\lim_{s\to 0}\frac{2s}{s-5}=0$$

  - **But the final value theorem is not valid because**

    $$\lim_{t\to\infty} f(t)=\lim_{t\to\infty} 2e^{5t}=\infty$$

EXAMPLE ON LAPLACE TRANSFORM (2)

- **What is the final value of the following system?**

$$x'' + x' + x = \sin t;\; x(0) = x'(0) = 0$$

$$\Rightarrow\; s^2 X(s) + sX(s) + X = \frac{1}{s^2+1} \quad \Rightarrow\; \text{x(s)=} \frac{1}{(s^2+1)(s^2+s+1)}$$

$$x(\infty) = \lim_{s\to 0} \frac{s}{(s^2+1)(s^2+s+1)} = 0$$

  – **Actually,** $x(\infty)$ **cannot be defined due to sin** t **term.**

- **Find the Laplace transform for** $(t\sin wt)$ **?**

$$\text{From } \frac{dF(s)}{ds} = \mathcal{L}\left[-t \cdot f(t)\right]$$

$$\mathcal{L}\left[t \cdot \sin wt\right] = -\frac{d}{ds}\left[\frac{w}{s^2+w^2}\right] = \frac{2ws}{(s^2+w^2)^2}$$

INVERSE LAPLACE TRANSFORM

- **Used to recover the solution in time domain**

$$\mathcal{L}^{-1}\left\{F(s)\right\} = f(t)$$

  - **From the table**
  - **By partial fraction expansion**
  - **By inversion using contour integral**

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{2\pi j}\oint_C e^{st}F(s)\,ds$$

- **Partial fraction expansion**
  - **After the partial fraction expansion, it requires to know some simple formula of inverse Laplace transform such as**

$$\frac{1}{(\tau s+1)},\ \frac{s}{(s+\beta)^2+\omega^2},\ \frac{(n-1)!}{s^n},\ \frac{e^{-\theta s}}{\tau^2 s^2+2\zeta\tau s+1},\ \text{etc.}$$

PARTIAL FRACTION EXPANSION

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+p_1)\cdots(s+p_n)}=\frac{a_1}{(s+p_1)}+\cdots+\frac{a_n}{(s+p_n)}$$

- **Case I: All p_i' s are distinct and real**
  - **By a root-finding technique, find all roots (time-consuming)**
  - **Find the coefficients for each fraction**
    - **Comparison of the coefficients after multiplying the denominator**
    - **Replace some values for s and solve linear algebraic equation**
    - **Use of Heaviside expansion**
      - **Multiply both side by a factor, $(s+p_i)$, and replace s with $-p_i$.**

$$a_i=(s+p_i)\frac{N(s)}{D(s)}\bigg|_{s=-p_i}$$

  - **Inverse LT:**

$$f(t)=a_1e^{-p_1t}+a_2e^{-p_2t}+\cdots+a_ne^{-p_nt}$$

- **Case II: Some roots are repeated**

$$F(s)=\frac{N(s)}{D(s)}=\frac{N(s)}{(s+p)^r}=\frac{b_{r-1}s^{r-1}+\cdots+b_0}{(s+p)^r}=\frac{a_1}{(s+p)}+\cdots+\frac{a_r}{(s+p)^r}$$

  - **Each repeated factors have to be separated first.**
  - **Same methods as Case I can be applied.**
  - **Heaviside expansion for repeated factors**

$$a_{r-i}=\frac{1}{i!}\frac{d^{(i)}}{ds^{(i)}}\left(\frac{N(s)}{D(s)}(s+p)^r\right)\bigg|_{s=-p}\qquad (i=0,\cdots,r-1)$$

  - **Inverse LT**

$$f(t)=a_1e^{-pt}+a_2te^{-pt}+\cdots+\frac{a_r}{(r-1)!}t^{r-1}e^{-pt}$$

- **Case III: Some roots are complex**

$$F(s)=\frac{N(s)}{D(s)}=\frac{c_1 s+c_0}{s^2+d_1 s+d_0}=\frac{a_1(s+b)+b_1 w}{(s+b)^2+w^2}$$

- **Each repeated factors have to be separated first.**
- **Then,**

$$\frac{a_1(s+b)+b_1 w}{(s+b)^2+w^2}=a_1\frac{(s+b)}{(s+b)^2+w^2}+b_1\frac{w}{(s+b)^2+w^2}$$

where $b=d_1/2,\ \ w=\sqrt{d_0-d_1^2/4}$

$$a_1=c_1,\quad b_1=\left(c_0-a_1 b\right)/w$$

- **Inverse LT**

$$f(t)=a_1 e^{-bt}\cos wt+b_1 e^{-bt}\sin wt$$

EXAMPLES ON INVERSE LAPLACE TRANSFORM

- $$F(s)=\frac{(s+5)}{s(s+1)(s+2)(s+3)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}+\frac{D}{s+3} \quad \text{(distinct)}$$
  - **Multiply each factor and insert the zero value**

$$\left.\frac{(s+5)}{(s+1)(s+2)(s+3)}\right|_{s=0}=\left.\left(A+s\frac{B}{s+1}+s\frac{C}{s+2}+s\frac{D}{s+3}\right)\right|_{s=0} \Rightarrow A=5/6$$

$$\left.\frac{(s+5)}{s(s+2)(s+3)}\right|_{s=-1}=\left.\left(\frac{A(s+1)}{s}+B+\frac{C(s+1)}{s+2}+\frac{D(s+1)}{s+3}\right)\right|_{s=-1} \Rightarrow B=-2$$

$$\left.\frac{(s+5)}{s(s+1)(s+3)}\right|_{s=-2}=\left.\left(\frac{A(s+2)}{s}+\frac{B(s+2)}{s+1}+C+\frac{D(s+2)}{s+3}\right)\right|_{s=-2} \Rightarrow C=3/2$$

$$\left.\frac{(s+5)}{s(s+1)(s+2)}\right|_{s=-3}=\left.\left(\frac{A(s+3)}{s}+\frac{B(s+3)}{s+1}+\frac{C(s+3)}{s+2}+D\right)\right|_{s=-3} \Rightarrow D=-1/3$$

$$\therefore f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{5}{6}-2e^{-t}+\frac{3}{2}e^{-2t}-\frac{1}{3}e^{-3t}$$

- $$F(s)=\frac{1}{(s+1)^3(s+2)}=\frac{As^2+Bs+C}{(s+1)^3}+\frac{D}{(s+2)} \quad \text{(repeated)}$$

$$1=(As^2+Bs+C)(s+2)+D(s+1)^3$$
$$=(A+D)s^3+(2A+B+3D)s^2+(2B+C+3D)s+(2C+D)$$
$$\therefore A=-D,\ \ 2A+B+3D=0,\ \ 2B+C+3D=0,\ \ 2C+D=1$$
$$\Rightarrow A=1,\ \ B=1,\ \ C=1,\ \ D=-1$$

  - **Use of Heaviside expansion** $a_{r-i}=\frac{1}{i!}\frac{d^{(i)}}{ds^{(i)}}\left(\frac{N(s)}{D(s)}(s+p)^r\right)\Big|_{s=-p} \quad (i=0,\cdots,r-1)$

$$\frac{s^2+s+1}{(s+1)^3}=\frac{a_1}{(s+1)}+\frac{a_2}{(s+1)^2}+\frac{a_3}{(s+1)^3}$$

$$(i=0):\ a_3=\left(s^2+s+1\right)\Big|_{s=-1}=1$$
$$(i=1):\ a_2=\frac{1}{1!}\frac{d}{ds}\left(s^2+s+1\right)\Big|_{s=-1}=-1$$
$$(i=2):\ a_1=\frac{1}{2!}\frac{d^2}{ds^2}\left(s^2+s+1\right)\Big|_{s=-1}=1$$

$$\therefore f(t)=\mathrm{L}^{-1}\left\{F(s)\right\}=e^{-t}-te^{-t}+\frac{1}{2}t^2e^{-t}-e^{-2t}$$

- $$F(s)=\frac{(s+1)}{s^2(s^2+4s+5)}=\frac{A(s+2)+B}{(s+2)^2+1}+\frac{Cs+D}{s^2} \quad \text{(complex)}$$

$$s+1=A(s+2)s^2+Bs^2+(Cs+D)(s^2+4s+5)$$

$$=(A+C)s^3+(2A+B+4C+D)s^2+(5C+4D)s+5D$$

$$\therefore A=-C,\ \ 2A+B+4C+D=0,\ \ 5C+4D=1,\ \ 5D=1$$

$$\Rightarrow A=-1/25,\ B=-7/25,\ C=1/25,\ D=1/5$$

$$\frac{A(s+2)+B}{(s+2)^2+1}=-\frac{1}{25}\frac{(s+2)}{(s+2)^2+1}-\frac{7}{25}\frac{B}{(s+2)^2+1}$$

$$\frac{Cs+D}{s^2}=\frac{1}{25}\frac{1}{s}+\frac{1}{5}\frac{1}{s^2}$$

$$\therefore f(t)=\mathcal{L}^{-1}\{F(s)\}=-\frac{1}{25}e^{-2t}\cos t-\frac{7}{25}e^{-2t}\sin t+\frac{1}{25}+\frac{1}{5}t$$

- $$F(s)=\frac{1+e^{-2s}}{(4s+1)(3s+1)}=\left(\frac{A}{4s+1}+\frac{B}{3s+1}\right)(1+e^{-2s})\quad \text{(Time delay)}$$

$$A=1/(3s+1)\big|_{s=-1/4}=4,\quad B=1/(4s+1)\big|_{s=-1/3}=-3$$

$$\therefore f(t)=\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left\{\frac{4}{4s+1}-\frac{3}{3s+1}\right\}+\mathcal{L}^{-1}\left\{\frac{4e^{-2s}}{4s+1}-\frac{3e^{-2s}}{3s+1}\right\}$$

$$=e^{-t/4}-e^{-t/3}+\left(e^{-(t-2)/4}-e^{-(t-2)/3}\right)S(t-2)$$

It is a brain teaser!!!
But you have to live with it.

Hang on!!!
You are half way there.

SOLVING ODE BY LAPLACE TRANSFORM

- **Procedure**
  1. **Given linear ODE with initial condition,**
  2. **Take Laplace transform and solve for output**
  3. **Inverse Laplace transform**

- **Example:** Solve for $5\frac{dy}{dt} + 4y = 2;\ y(0) = 1$

$$\mathcal{L}\left\{5\frac{dy}{dt}\right\} + \mathcal{L}\{4y\} = \mathcal{L}\{2\} \quad \Rightarrow \quad 5(sY(s) - y(0)) + 4Y(s) = \frac{2}{s}$$

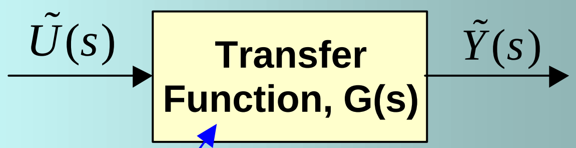
$$(5s+4)Y(s) = \frac{2}{s} + 5 \quad \Rightarrow \quad Y(s) = \frac{5s+2}{s(5s+4)}$$

$$\therefore\ y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{0.5}{s} + \frac{2.5}{5s+4}\right\} = 0.5 + 0.5e^{-0.8t}$$

TRANSFER FUNCTION (1)

- **Definition**
  - **An algebraic expression for the dynamic relation between the input and output of the process model**

$$5\frac{dy}{dt}+4y=u;\ y(0)=1$$



Let $\tilde{y}=y-1$ and $\tilde{u}=u-4$

$$(5s+4)\tilde{Y}(s)=\tilde{U}(s)\ \Rightarrow\ \frac{\tilde{Y}(s)}{\tilde{U}(s)}=\frac{1}{5s+4}=\frac{0.25}{1.25s+1}=G(s)$$

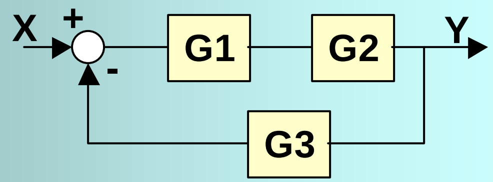
- **How to find transfer function**
  1. **Find the equilibrium point**
  2. **If the system is nonlinear, then linearize around equil. point**
  3. **Introduce deviation variables**
  4. **Take Laplace transform and solve for output**
  5. **Do the Inverse Laplace transform and recover the original variables from deviation variables**

TRANSFER FUNCTION (2)

- **Benefits**
  - **Once TF is known, the output response to various given inputs can be obtained easily.**

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)U(s)\} \neq \mathcal{L}^{-1}\{G(s)\}\,\mathcal{L}^{-1}\{U(s)\}$$

  - **Interconnected system can be analyzed easily.**
    - **By block diagram algebra**



$$\frac{Y(s)}{X(s)} = \frac{G1(s)G2(s)}{1 + G1(s)G2(s)G3(s)}$$

  - **Easy to analyze the qualitative behavior of a process, such as stability, speed of response, oscillation, etc.**
    - **By inspecting "Poles" and "Zeros"**
    - **Poles: all *s*' s satisfying $D(s)=0$**
    - **Zeros: all *s*' s satisfying $N(s)=0$**

TRANSFER FUNCTION (3)

- **Steady-state Gain: The ratio between ultimate changes in input and output**

$$\text{Gain}=K=\frac{\Delta\text{ouput}}{\Delta\text{input}}=\frac{(y(\infty)-y(0))}{(u(\infty)-u(0))}$$

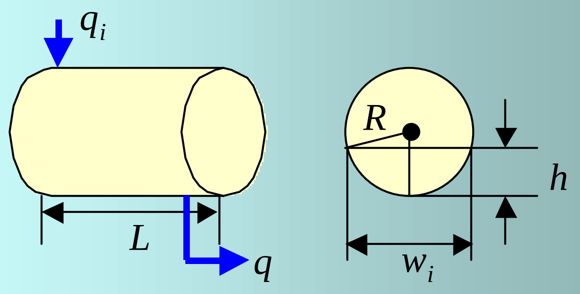
  - **For a unit step change in input, the gain is the change in output**
  - **Gain may not be definable: for example, integrating processes and processes with sustaining oscillation in output**
  - **From the final value theorem, unit step change in input with zero initial condition gives**

$$K=\frac{y(\infty)}{1}=\lim_{s\to 0} sY(s)=\lim_{s\to 0} sG(s)\frac{1}{s}=\lim_{s\to 0} G(s)$$

  - **The transfer function itself is an impulse response of the system** $Y(s)=G(s)U(s)=G(s)\mathcal{L}\{\delta(t)\}=G(s)$

EXAMPLE

- **Horizontal cylindrical storage tank (Ex4.7)**



$$\frac{dm}{dt} = r\frac{dV}{dt} = r q_i - r q$$

$$V(h) = \int_0^h L w_i(\tilde{h})d\tilde{h} \Rightarrow \frac{dV}{dt} = L w_i(h)\frac{dh}{dt}$$

$$w_i(h)/2 = \sqrt{R^2 + (R-h)^2} = \sqrt{(2R-h)h}$$

$$w_i L\frac{dh}{dt} = q_i - q \quad \Rightarrow \quad \frac{dh}{dt} = \frac{1}{2L\sqrt{(D-h)h}}(q_i - q) \quad \text{(Nonlinear ODE)}$$

- **Equilibrium point:** $(\bar{q}_i, \bar{q}, \bar{h})$ $\qquad 0 = (\bar{q}_i - \bar{q})/(2L\sqrt{(D-\bar{h})\bar{h}})$
  (if $\bar{q}_i = \bar{q}$, $\bar{h}$ can be any value in $0 \le \bar{h} \le D$.)
- **Linearization:**

$$\frac{dh}{dt} = f(h, q_i, q) = \left.\frac{\partial f}{\partial h}\right|_{(\bar{h},\bar{q}_i,\bar{q})}(h-\bar{h}) + \left.\frac{\partial f}{\partial q_i}\right|_{(\bar{h},\bar{q}_i,\bar{q})}(q_i - \bar{q}_i) + \left.\frac{\partial f}{\partial q}\right|_{(\bar{h},\bar{q}_i,\bar{q})}(q-\bar{q})$$

$$\left.\frac{\partial f}{\partial h}\right|_{(\bar{h},\bar{q}_i,\bar{q})} = (\bar{q}_i - \bar{q})\frac{\partial}{\partial h}\frac{-1}{2L\sqrt{(D-h)h}} = 0 \quad (\because \bar{q}_i = \bar{q})$$

$$\left.\frac{\partial f}{\partial q}\right|_{(\bar{h},\bar{q}_i,\bar{q})} = \frac{-1}{2L\sqrt{(D-\bar{h})\bar{h}}}, \qquad \left.\frac{\partial f}{\partial q_i}\right|_{(\bar{h},\bar{q}_i,\bar{q})} = \frac{1}{2L\sqrt{(D-\bar{h})\bar{h}}}$$

Let this term be k

$$s\tilde{H}(s) = k\tilde{Q}_i(s) - k\tilde{Q}(s)$$

- **Transfer function between** $\tilde{H}(s)$ **and** $\tilde{Q}(s)$: $-\frac{k}{s}$ **(integrating)**
- **Transfer function between** $\tilde{H}(s)$ **and** $\tilde{Q}_i(s)$: $\frac{k}{s}$ **(integrating)**
- **If** $\bar{h}$ **is near 0 or** D, k **becomes very large and** $\bar{h}$ **is around** $\bar{h}/2$ **,** k **becomes minimum.**

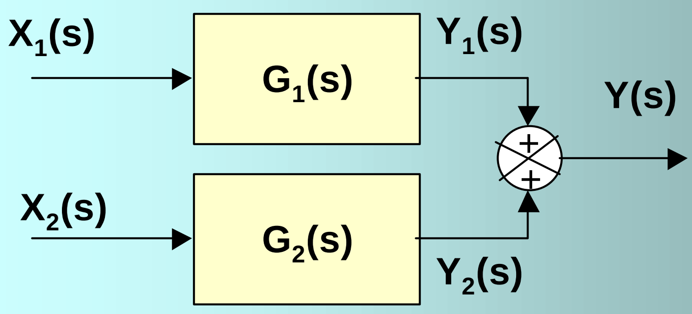
Þ **The model could be quite different depending on the operating condition used for the linearization.**

Þ **The best suitable range for the linearization in this case is around** $\bar{h}/2$ **. (less change in gain)**

Þ **Linearized model would be valid in very narrow range near 0**

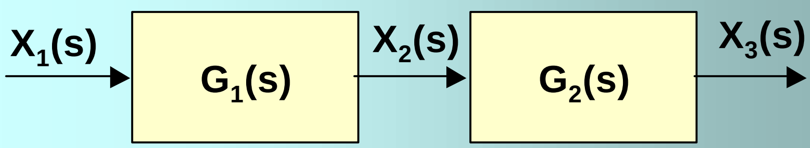
PROPERTIES OF TRANSFER FUNCTION

- **Additive property**



$$Y(s) = Y_1(s) + Y_2(s)$$
$$= G_1(s)X_1(s) + G_2(s)X_2(s)$$

- **Multiplicative property**



$$X_3(s) = G_2(s)X_2(s)$$
$$= G_2(s)\left[G_1(s)X_1(s)\right] = G_2(s)G_1(s)X_1(s)$$

- **Physical realizability**
  - **In a transfer function, the order of numerator(m) is greater than that of denominator(n): called "physically unrealizable"**
  - **The order of derivative for the input is higher than that of output. (requires future input values for current output)**

EXAMPLES ON TWO TANK SYSTEM

- **Two tanks in series (Ex3.7)**
  - **No reaction**

$$V_1 \frac{dc_1}{dt} + qc_1 = qc_i$$

$$V_2 \frac{dc_2}{dt} + qc_2 = qc_1$$

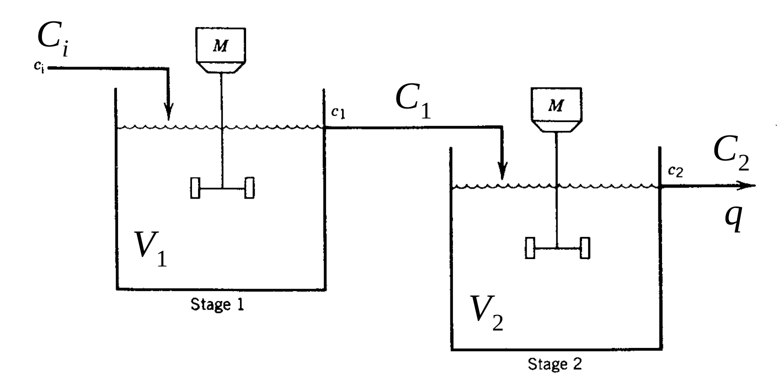


Figure 3.4. Two-stage stirred-tank reactor system.

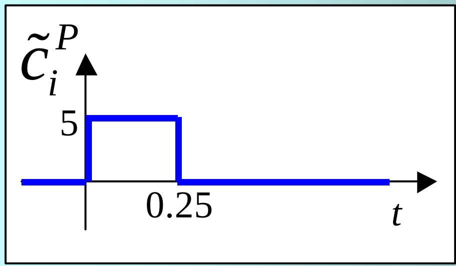
  - **Initial condition:** $c_1(0) = c_2(0) = 1$ kg mol/m^3 **(Use deviation var.)**
  - **Parameters:** V_1/q**=2 min.,** V_2/q**=1.5 min.**
  - **Transfer functions**

$$\frac{\tilde{C}_1(s)}{\tilde{C}_i(s)} = \frac{1}{(V_1/q)s+1} \qquad \frac{\tilde{C}_2(s)}{\tilde{C}_1(s)} = \frac{1}{(V_2/q)s+1}$$

$$\frac{\tilde{C}_2(s)}{\tilde{C}_i(s)} = \frac{\tilde{C}_2(s)}{\tilde{C}_1(s)} \frac{\tilde{C}_1(s)}{\tilde{C}_i(s)} = \frac{1}{\left((V_2/q)s+1\right)\left((V_1/q)s+1\right)}$$

- **Pulse input**

$$\tilde{C}_i^P(s) = \frac{5}{s}(1 - e^{-0.25s})$$



- **Equivalent impulse input**

$$\tilde{C}_i^d(s) = \mathcal{L}\{(5 \times 0.15)\delta(t)\} = 1.25$$

- **Pulse response vs. Impulse response**

$$\tilde{C}_1^P(s) = \frac{1}{2s+1}\tilde{C}_i^P(s) = \frac{5}{s(2s+1)}(1 - e^{-0.25s})$$

$$= \left(\frac{5}{s} - \frac{10}{2s+1}\right)(1 - e^{-0.25s})$$

$$\Rightarrow \tilde{c}_1^P(t) = 5(1 - e^{-t/2}) - 5(1 - e^{-(t-0.25)/2})S(t - 0.25)$$

$$\tilde{C}_1^d(s) = \frac{1}{2s+1}\tilde{C}_i^d(s) = \frac{1.25}{(2s+1)}$$

$$\Rightarrow \tilde{c}_1^d = 0.625e^{-t/2}$$

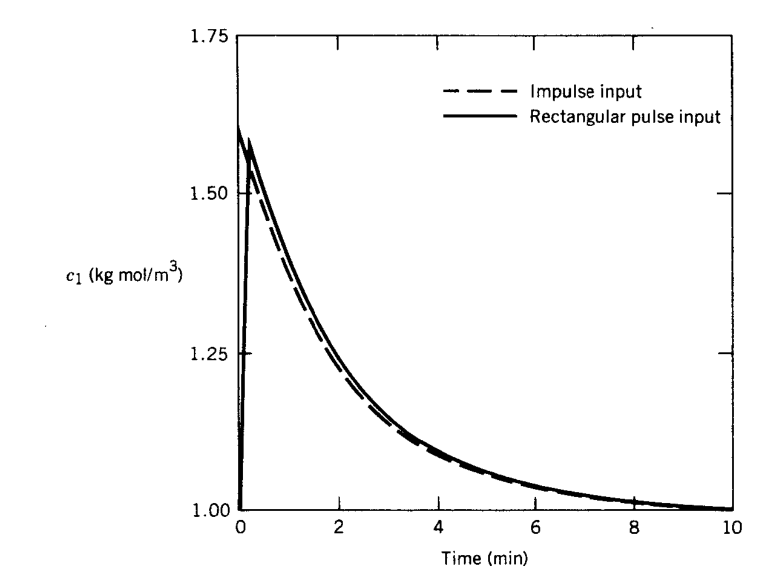


Figure 3.6. Reactor Stage 1 response.

$$\tilde{C}_2^P(s) = \frac{1}{(2s+1)(1.5s+1)}\tilde{C}_i^P(s) = \frac{5}{s(2s+1)(1.5s+1)}(1-e^{-0.25s})$$

$$= \left(\frac{5}{s} - \frac{40}{2s+1} + \frac{22.5}{1.5s+1}\right)(1-e^{-0.25s})$$

$$\Rightarrow \tilde{c}_2^P(t) = (5 - 20e^{-t/2} + 15e^{-t/1.5}) - (5 - 20e^{-(t-0.25)/2} + 15e^{-(t-0.25)/1.5})S(t-0.25)$$

$$\tilde{C}_2^d(s) = \frac{1}{(2s+1)(1.5s+1)}\tilde{C}_i^d(s)$$

$$= \frac{1.25}{(2s+1)(1.5s+1)}$$

$$= \frac{5}{2s+1} - \frac{3.75}{1.5s+1}$$

$$\Rightarrow \tilde{c}_2^d = 2.5e^{-t/2} - 2.5e^{-t/1.5}$$

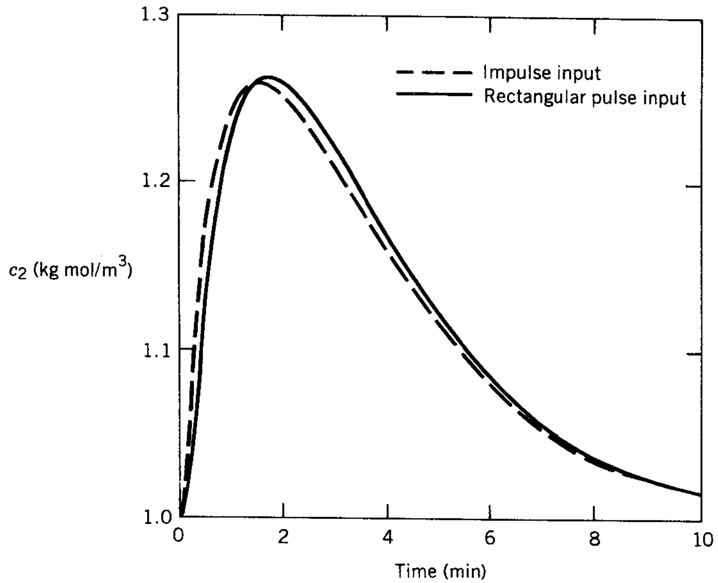


Figure 3.7. Reactor Stage 2 response.