Subsections

Modeling, Computers, and Error Analysis

Mathematical Modeling and Engineering Problem-Solving

A Simple Mathematical Model

수학적 모델이라는 것은 수학적 용어로서 물리적 자연현상의 중요한 부분을 식으로서 구성하는 것이다.

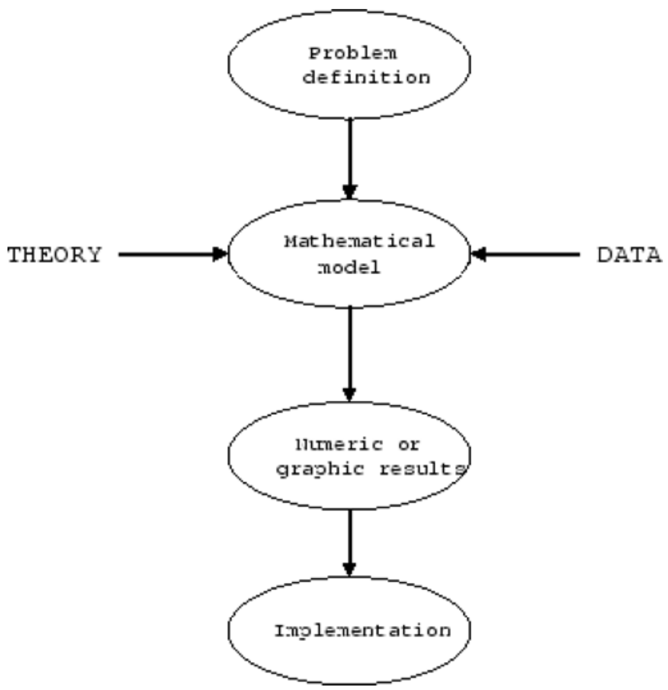

Figure 1.1: The engineering problem-solving process.

이러한 수학모델을에서 결과를 얻는 방법은 크게 두가지가 있다.

- Analytical or exact solution
- Numerical solution

위 두가지 가운데 해석적 해를 얻는 방법은 다룰 수 있는 문제가 한정되어 있고 또한 대부분의 실제 문제는 비선형적이고 복잡한 과정이 포함되어 있으므로 적절하지 않다. 그러므로 수치적 해를 구할 수 밖에 없다.

Computers and Software

The Software Development Process

- Programming Style
- Modular Design : Devide into small subprograms
- Top-down Design : Sysmtematic development process
- Structured Programming : How the actual program code is developed

Algorithm Design

- Flowschart : a visual or graphical representation of an algorithm
- Pseudocode : bridges the gap between flowcharts and computer code

Program Composition

- High-level and Macro Languages : C, Fortran, Basic
- Structured Programming
  - consist of the three funcdamental control structures of sequence, selection, and repetition
  - only one entrance and one exit
  - Unconditional transfers should be avoided
  - identified with comments and visual devices such as indentation, blank lines, and blank spaces

Quality Control

- Errors or ``Bugs''
  - Syntax errors
  - Link or build errors
  - Run-time errors
  - Logic errors
- Debugging
- Testing

Approximations and Round-Off Errors

Significant Figures

Significant figure : The reliability of a numerical value

Accuracy and Precision

- Accuracy : How closely a computed or measured value agrees with the true value
- Precision : How closely individual computed or measured values agree with each other

Error Definitions

- Truncation error : approximations are used to represent exact mathematical prodedures
- Round-off error : numbers having limited significant figures are used to represent exact numbers

See Figure 3.10, 3.11 and 3.12 in the textbook.

Truncation Errors and the Taylor Series

The Taylor Series

If a function $f(x)$ can be represent by a power series on the interval $(-a, a)$, then the function has derivative of all orders on that interval and the power series is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \tag{1.1}$$

and this power-series expansion of $f(x)$ about the origin is called a Maclaurin series.

If the expansion is about the point $x = a$, we have the Taylor series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(0)}{3!}(x-a)^3 + \dots \tag{1.2}$$

Taylor series specifies the value of function at one point, x, in terms of the value of the function and its derivatives at a reference point, a. It is occasionally useful to express a Taylor series in a notation that show how the function behaves at a distance h from a fixed point a. If we call $x = a + h$ in the preceding series, so that $x - a = h$, we get

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(0)}{3!}h^3 + \dots \tag{1.3}$$

Or with the substitution $a + h \rightarrow x_{i+1}$ and $a \rightarrow x_i$ we have an alternate form

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \dots + \frac{f^{(n)}(x_i)}{n!}(x_{i+1} - x_i)^n + R_n \tag{1.4}$$

R_n term is a reminder term to account for all terms from $n + 1$ to infinity:

$$R_n = \frac{f^{(n+1)}(\zeta)}{(n+1)!}(x_{i+1} - x_i)^{n+1} \tag{1.5}$$

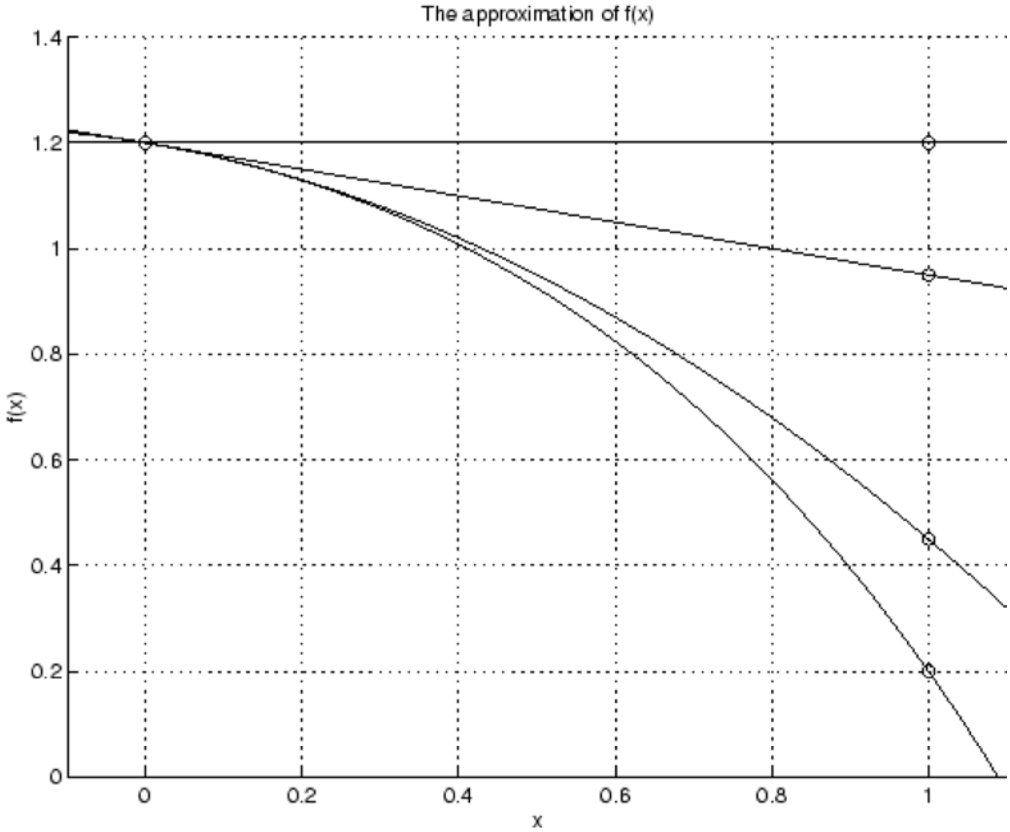

Figure 1.2: The approximation of $f(x)$ with various order of Taylor series.

- Mean-value theorem:
  If a function $f(x)$ and its first derivative are continuous over an interval from x_i and x_{i+1}, then there exists at least one point on the function that has a slope, designated by $f'(\xi)$, that is parallel to the line joining $f(x_i)$ and $f(x_{i+1})$.

See Figure 4.3 in the textbook.

Using the Taylor Series to Estimate Truncation Errors

Taylor series expansion of $v(t)$:

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + \frac{v''(t_i)}{2!}(t_{i+1} - t_i)^2 + \cdots + R_n \tag{1.6}$$

Truncate the series after the first derivative term:

$$v(t_{i+1}) = v(t_i) + v'(t_i)(t_{i+1} - t_i) + R_1 \tag{1.7}$$

And

$$v'(t_i) = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} - \frac{R_1}{t_{i+1} - t_i} \tag{1.8}$$

Truncation error is

$$\frac{R_1}{t_{i+1}-t_i} = \frac{v''(\xi)}{2!}(t_{i+1}-t_i) \tag{1.9}$$

or

$$\frac{R_1}{t_{i+1}-t_i} = O(t_{i+1}-t_i) \tag{1.10}$$

The error of our derivative approximation should be proportional to the step size. Consequently, if we halve the step size, we would expect to halve the error of the derivative.

Numerical Differentiation

- Forward Difference Approximation

  $$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + O(t_{i+1} - t_i)$$

  or

  $$f'(x_i) = \frac{\Delta f_i}{h} + O(h)$$

  where Δf_i is the first forward difference.

- Backward Difference Approximation

  $$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(t_i - t_{i-1})$$

  or

  $$f'(x_i) = \frac{\nabla f_i}{h} + O(h)$$

- Central Difference Approximation

  $$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} + O(h^2)$$

  Notice that the truncation error is of the order of h^2 in contrast to the forward and backward approximations that were of the order of h. Consequently, the Taylor series analysis yields the practical information that the centered difference is a more accurate representation of the derivative.

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