7. Microscopic Balances

7.1 Introduction

Macroscopic balances : averages over surfaces normal to the mean flow direction, no detailed information (fluid motion and forces on a fine scale).

Microscopic balances : apply conservation eq'ns on a differential control volume ⇒ obtain a set of differential eq'ns with position (x,y,z) and time (t) as independent variables. ⇒ Integration of these eq'ns provide a complete description of the flow at each point.

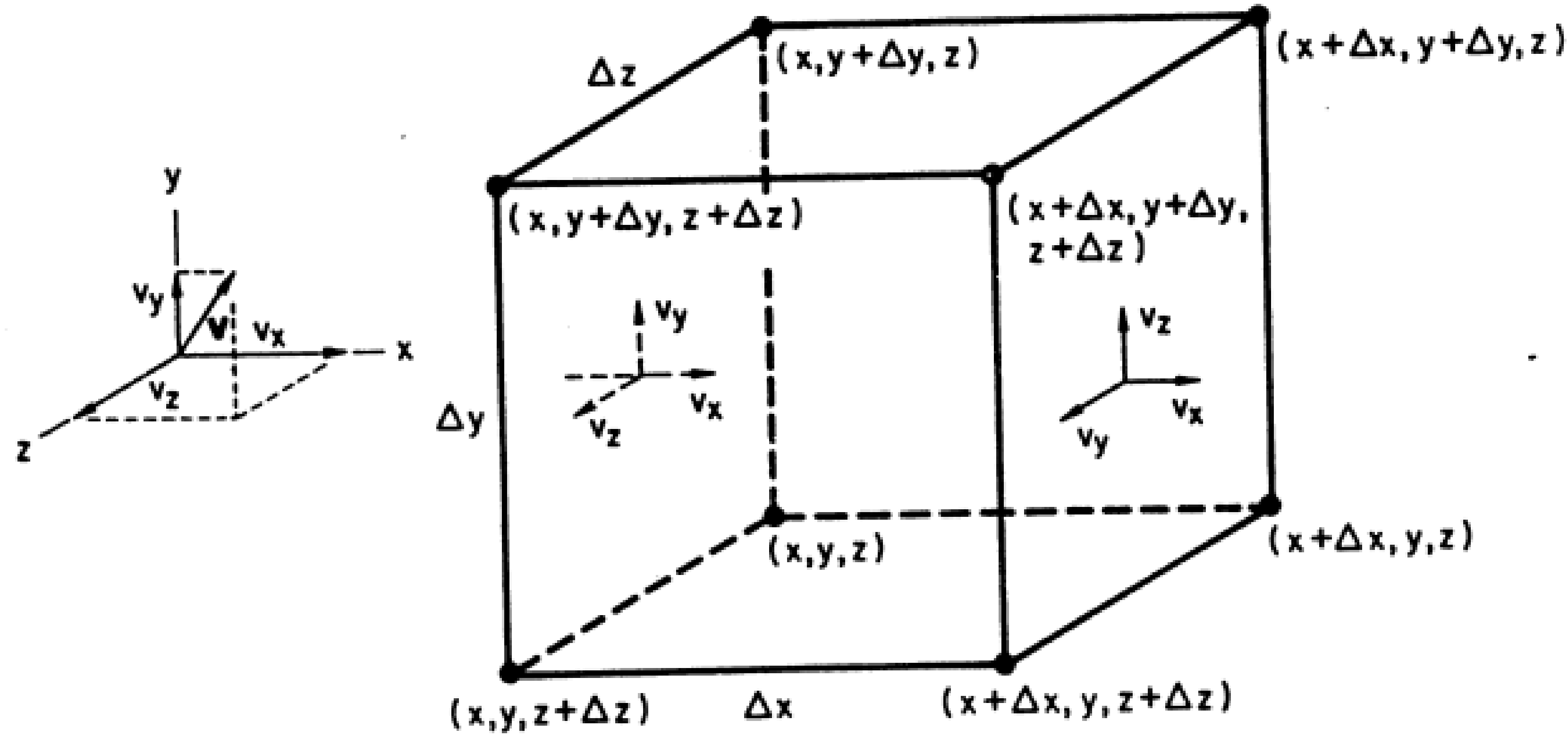

Fig. 7-2. Cubic control volume.

7.2 Conservation of Mass

Continuity eq'n:

Control volume: small cube having faces of length $\Delta x, \Delta y, \Delta z$ with one corner at position (x, y, z), total volume is $\Delta x \Delta y \Delta z$.

Velocity vector $\boldsymbol{\upsilon} = (\upsilon_x, \upsilon_y, \upsilon_z)$

Conservation of mass:

the rate of change of mass in the control volume
= the rate at which mass enters the control volume
− the rate at which mass leaves the control volume

the rate of change of mass in the control volume

$$= \frac{\partial}{\partial t} \bar{\rho} \Delta x \Delta y \Delta z$$

the rate at which mass enters the control volume

$$= \langle \rho v_x \rangle \Delta y \Delta z \,|_{x} + \langle \rho v_y \rangle \Delta x \Delta z \,|_{y} + \langle \rho v_z \rangle \Delta x \Delta y \,|_{z}$$

the rate at which mass leaves the control volume

$$= \langle \rho v_x \rangle \Delta y \Delta z \,|_{x+\Delta x} + \langle \rho v_y \rangle \Delta x \Delta z \,|_{y+\Delta y} + \langle \rho v_z \rangle \Delta x \Delta y \,|_{z+\Delta z}$$

The conservation of mass is then

$$\frac{\partial \bar{\rho}}{\partial t} \Delta x \Delta y \Delta z = \langle \rho v_x \rangle \Delta y \Delta z \,|_{x} + \langle \rho v_y \rangle \Delta x \Delta z \,|_{y} + \langle \rho v_z \rangle \Delta x \Delta y \,|_{z} - \langle \rho v_x \rangle \Delta y \Delta z \,|_{x+\Delta x} - \langle \rho v_y \rangle \Delta x \Delta z \,|_{y+\Delta y} - \langle \rho v_z \rangle \Delta x \Delta y \,|_{z+\Delta z}$$

Dividing by the volume $\Delta x \Delta y \Delta z$,

$$\frac{\partial \bar{\rho}}{\partial t} = -\frac{\langle \rho v_x \rangle \,|_{x+\Delta x} - \langle \rho v_x \rangle \,|_{x}}{\Delta x} - \frac{\langle \rho v_y \rangle \,|_{y+\Delta y} - \langle \rho v_y \rangle \,|_{y}}{\Delta y} - \frac{\langle \rho v_z \rangle \,|_{z+\Delta z} - \langle \rho v_z \rangle \,|_{z}}{\Delta z}$$

Taking the limit of $\Delta x \to 0, \Delta y \to 0, \Delta z \to 0$,

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\langle \rho \upsilon_x \rangle |_{x+\Delta x} - \langle \rho \upsilon_x \rangle |_x}{\Delta x} = \frac{\partial (\rho \upsilon_x)}{\partial x}$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\langle \rho \upsilon_y \rangle |_{y+\Delta y} - \langle \rho \upsilon_y \rangle |_y}{\Delta y} = \frac{\partial (\rho \upsilon_y)}{\partial y}$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\langle \rho \upsilon_z \rangle |_{z+\Delta z} - \langle \rho \upsilon_z \rangle |_z}{\Delta z} = \frac{\partial (\rho \upsilon_z)}{\partial z}$$

And $\overline{\rho} \to \rho$ as the volume shrinks to zero.

Thus we obtain

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho \upsilon_x}{\partial x} - \frac{\partial \rho \upsilon_y}{\partial y} - \frac{\partial \rho \upsilon_z}{\partial z}$$: continuity eq'n

or $$\frac{D\rho}{Dt} = -\rho\left(\frac{\partial \upsilon_x}{\partial x} + \frac{\partial \upsilon_y}{\partial y} + \frac{\partial \upsilon_z}{\partial z}\right)$$

where

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z}$$

: substantial derivative of ρ

Substantial derivative: the rate of change with time as recorded by an observer moving with a fluid particle

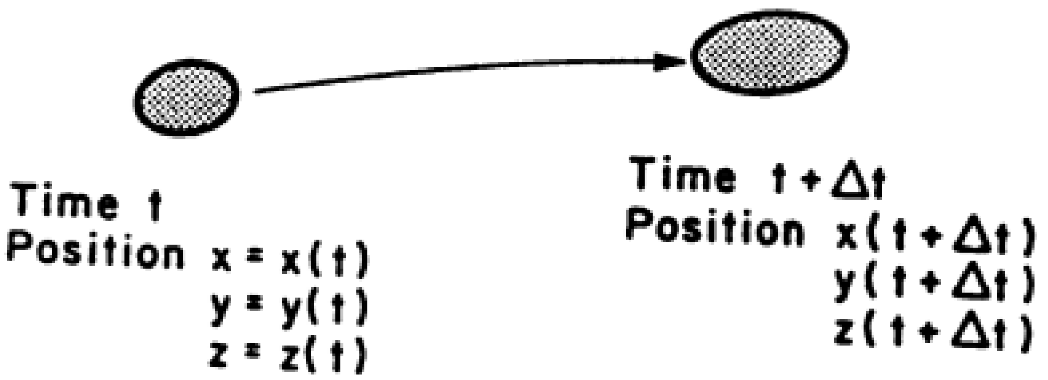

Fig. 7-3. A particle of fluid changes spacial position with time.

Vector notation:

The velocity vector in Cartesian system is written in terms of th unit vectors and components

$$\boldsymbol{v} = v_x \boldsymbol{i} + v_y \boldsymbol{j} + v_z \boldsymbol{k}$$

The gradient of a scalar :

$$\nabla \xi = \frac{\partial \xi}{\partial x} \boldsymbol{i} + \frac{\partial \xi}{\partial y} \boldsymbol{j} + \frac{\partial \xi}{\partial z} \boldsymbol{k}$$

The gradient operator : $\nabla = \frac{\partial}{\partial x} \boldsymbol{i} + \frac{\partial}{\partial y} \boldsymbol{j} + \frac{\partial}{\partial z} \boldsymbol{k}$

The dot products :

$$\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1 \quad , \qquad \boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0$$

Thus,

$$\boldsymbol{v} \cdot \nabla \xi = v_x \frac{\partial \xi}{\partial x} + v_y \frac{\partial \xi}{\partial y} + v_z \frac{\partial \xi}{\partial z}$$

The inner product between ∇ and $\boldsymbol{v}$ is

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$: Divergence of $\boldsymbol{v}$ (scalar)

The continuity eq'n is thus written as

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \boldsymbol{v} \qquad \text{or} \qquad \frac{D\rho}{Dt} = -\rho \nabla \cdot \boldsymbol{v}$$

$$\left(\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho \right)$$

Incompressible fluid: $\nabla \cdot \boldsymbol{v} = 0$

7.3 Conservation of Momentum

Momentum flow:

Conservation of the x component of linear momentum:

the rate of change of x momentum in the control volume
= rate at which x momentum flows into the control volume
− rate at which x momentum flows out of the control volume
+ sum of all forces acting on control volume in x-direction

the rate of change of x momentum in the control volume

$$= \frac{\partial}{\partial t}\overline{\rho v_x}\Delta x \Delta y \Delta z$$

rate at which x momentum flows into the control volume

$$= (\rho \upsilon_x)(\upsilon_x \Delta y \Delta z)\,|_{x} + (\rho \upsilon_x)(\upsilon_y \Delta x \Delta z)\,|_{y} + (\rho \upsilon_x)(\upsilon_z \Delta x \Delta y)\,|_{z}$$

rate at which x momentum flows out of the control volume

$$= (\rho \upsilon_x)(\upsilon_x \Delta y \Delta z)\,|_{x+\Delta x} + (\rho \upsilon_x)(\upsilon_y \Delta x \Delta z)\,|_{y+\Delta y} + (\rho \upsilon_x)(\upsilon_z \Delta x \Delta y)\,|_{z+\Delta z}$$

Then, the x component of the momentum eq'n is written

$$\frac{\partial}{\partial t}\overline{\rho \upsilon_x}\Delta x \Delta y \Delta z = +\rho \upsilon_x \upsilon_x \Delta y \Delta z\,|_{x} - \rho \upsilon_x \upsilon_x \Delta y \Delta z\,|_{x+\Delta x}$$
$$+\rho \upsilon_x \upsilon_y \Delta x \Delta z\,|_{y} - \rho \upsilon_x \upsilon_y \Delta x \Delta z\,|_{y+\Delta y}$$
$$+\rho \upsilon_x \upsilon_z \Delta x \Delta y\,|_{z} - \rho \upsilon_x \upsilon_z \Delta x \Delta y\,|_{z+\Delta z}$$

+ sum of all forces acting on the control volume in x-direction

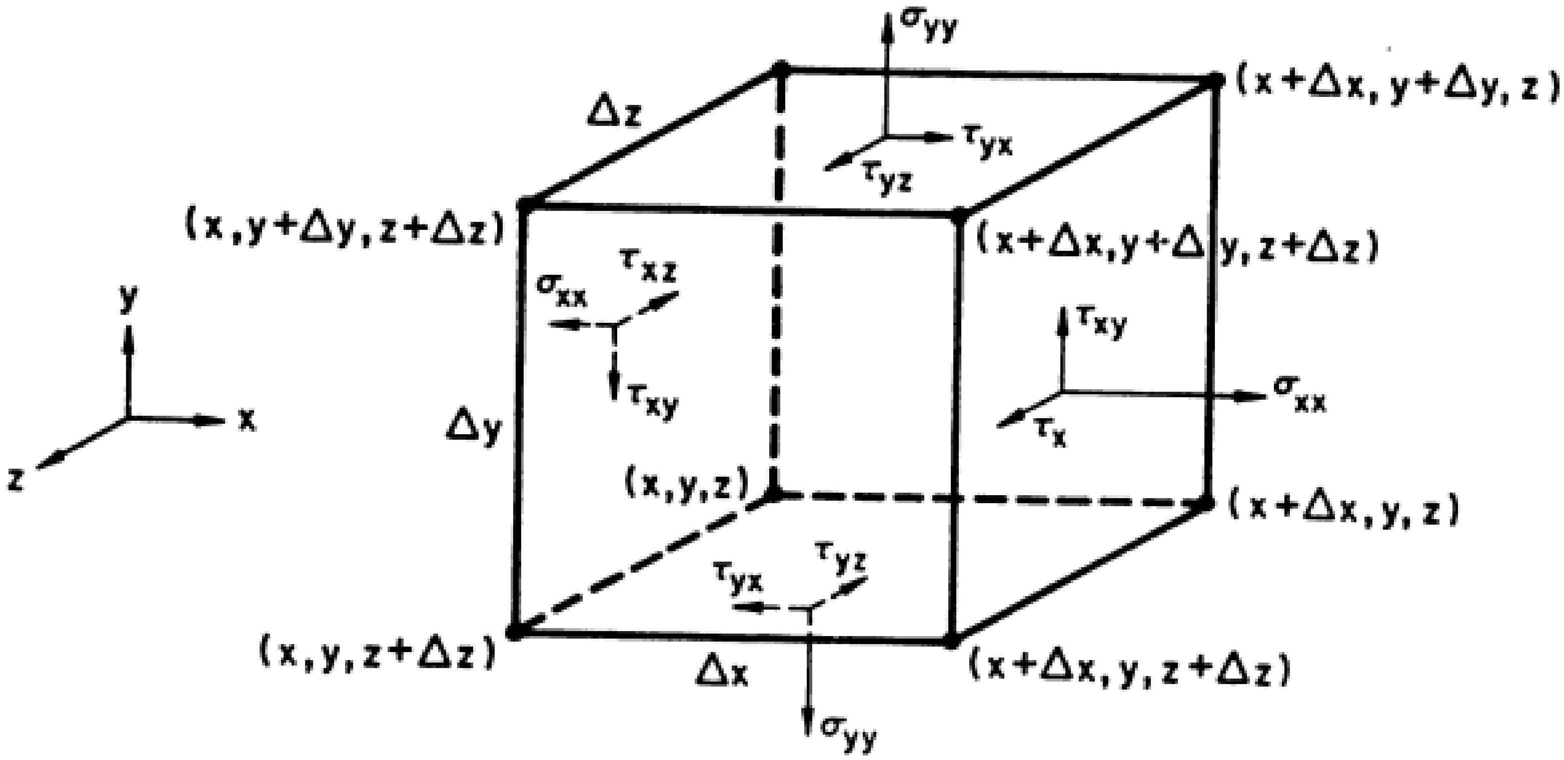

Fig. 7-4. Stresses acting on faces of the control volume.

Stress:

The force acting on the x face is $\boldsymbol{\sigma}_x \Delta y \Delta z$, and $\boldsymbol{\sigma}_x$ is resolved into its x, y, and z components

$$\boldsymbol{\sigma}_x = \sigma_{xx}\boldsymbol{i} + \tau_{xy}\boldsymbol{j} + \tau_{xz}\boldsymbol{k}$$

σ_{xx} : the stress component normal to the x face and represents tension or compression.

τ_{xy}, τ_{xz} : the stress components parallel to the x face and represents shear.

The first subscript, x : denotes the face.

The second subscript : denotes the direction in which the stress is acting.

Similarly, on the y and z planes,

$$\boldsymbol{\sigma}_y = \tau_{yx}\boldsymbol{i} + \sigma_{yy}\boldsymbol{j} + \tau_{yz}\boldsymbol{k}$$

$$\boldsymbol{\sigma}_z = \tau_{zx}\boldsymbol{i} + \tau_{zy}\boldsymbol{j} + \sigma_{zz}\boldsymbol{k}$$

Sign convention:

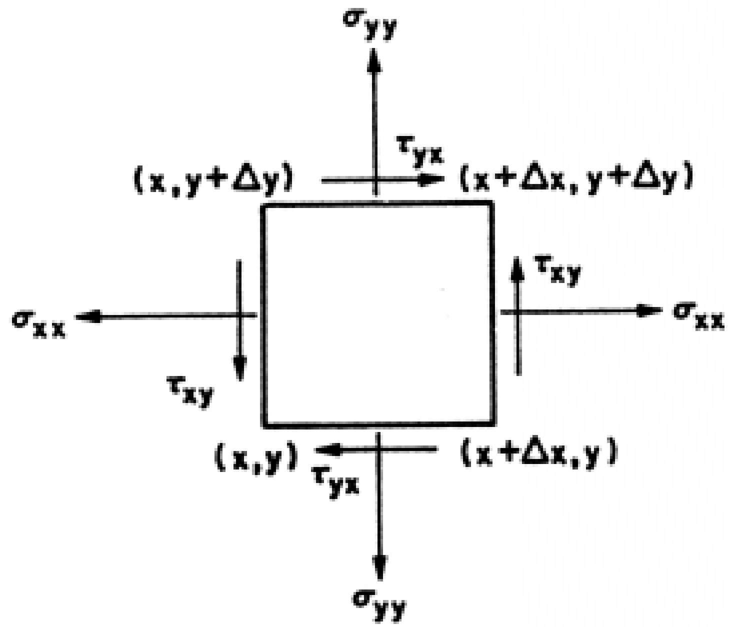

Fig. 7-5. Convention for positive and negative stresses on a face.

x-direction forces exerted by surrounding fluid on control volume:

$$\sigma_{xx}\Delta y\Delta z\mid_{x+\Delta x}-\sigma_{xx}\Delta y\Delta z\mid_{x}$$
$$+\tau_{yx}\Delta x\Delta z\mid_{y+\Delta y}-\tau_{yx}\Delta x\Delta z\mid_{y}$$
$$+\tau_{zx}\Delta x\Delta y\mid_{z+\Delta z}-\tau_{zx}\Delta x\Delta y\mid_{z}$$

Body force:

Body force in x direction = $\overline{\rho} g_x\Delta x\Delta y\Delta z$

Cauchy momentum eq'n:

The conservation of x momentum is

$$\frac{\partial}{\partial t}\overline{\rho v_x}\Delta x\Delta y\Delta z = +\rho v_x v_x\Delta y\Delta z\mid_{x}-\rho v_x v_x\Delta y\Delta z\mid_{x+\Delta x}$$
$$+\rho v_x v_y\Delta x\Delta z\mid_{y}-\rho v_x v_y\Delta x\Delta z\mid_{y+\Delta y}$$
$$+\rho v_x v_z\Delta x\Delta y\mid_{z}-\rho v_x v_z\Delta x\Delta y\mid_{z+\Delta z}$$
$$+\sigma_{xx}\Delta y\Delta z\mid_{x+\Delta x}-\sigma_{xx}\Delta y\Delta z\mid_{x}$$
$$+\tau_{yx}\Delta x\Delta z\mid_{y+\Delta y}-\tau_{yx}\Delta x\Delta z\mid_{y}$$
$$+\tau_{zx}\Delta x\Delta y\mid_{z+\Delta z}-\tau_{zx}\Delta x\Delta y\mid_{z}+\overline{\rho} g_x\Delta x\Delta y\Delta z$$

Dividing by the volume, $\Delta x \Delta y \Delta z$,

$$\frac{\partial \overline{\rho v_x}}{\partial t} + \frac{\rho v_x v_x|_{x+\Delta x} - \rho v_x v_x|_x}{\Delta x} + \frac{\rho v_y v_x|_{y+\Delta y} - \rho v_y v_x|_y}{\Delta y}$$

$$+ \frac{\rho v_z v_x|_{z+\Delta z} - \rho v_z v_x|_z}{\Delta z} = \frac{\sigma_{xx}|_{x+\Delta x} - \sigma_{xx}|_x}{\Delta x}$$

$$+ \frac{\tau_{yx}|_{y+\Delta y} - \tau_{yx}|_y}{\Delta y} + \frac{\tau_{zx}|_{z+\Delta z} - \tau_{zx}|_z}{\Delta z} + \overline{\rho} g_x$$

Taking the limit as $\Delta x, \Delta y, \Delta z \rightarrow 0$, we obtain

$$\frac{\partial \rho v_x}{\partial t} + \frac{\partial \rho v_x v_x}{\partial x} + \frac{\partial \rho v_y v_x}{\partial y} + \frac{\partial \rho v_z v_x}{\partial z}$$

$$= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x$$

The left-hand side is expanded into

$$\text{left-hand side} = +\rho \frac{\partial v_x}{\partial t} + \rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} + \rho v_z \frac{\partial v_x}{\partial z}$$

$$+ v_x \frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho v_x}{\partial x} + v_x \frac{\partial \rho v_y}{\partial y} + v_x \frac{\partial \rho v_z}{\partial z}$$

The second row becomes zero from continuity, and the final form of the x-momentum equation is

$$\rho \frac{Dv_x}{Dt} = \rho \frac{\partial v_x}{\partial t} + \rho v_x \frac{\partial v_x}{\partial x} + \rho v_y \frac{\partial v_x}{\partial y} + \rho v_z \frac{\partial v_x}{\partial z}$$
$$= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x$$

The y- and z-momentum equations are, respectively,

$$\rho \frac{Dv_y}{Dt} = \rho \frac{\partial v_y}{\partial t} + \rho v_x \frac{\partial v_y}{\partial x} + \rho v_y \frac{\partial v_y}{\partial y} + \rho v_z \frac{\partial v_y}{\partial z}$$
$$= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y$$

$$\rho \frac{Dv_z}{Dt} = \rho \frac{\partial v_z}{\partial t} + \rho v_x \frac{\partial v_z}{\partial x} + \rho v_y \frac{\partial v_z}{\partial y} + \rho v_z \frac{\partial v_z}{\partial z}$$
$$= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z$$

Stress symmetry:

$$\tau_{xy} = \tau_{yx}, \quad \tau_{yz} = \tau_{zy}, \quad \tau_{xz} = \tau_{zx}$$

from the conservation of angular momentum

The rate of change of moment of momentum

= the sum of the imposed torques

$$\frac{d}{dt}(\rho \Delta x \Delta y \Delta z r_g^2 \Omega)$$

$$= \frac{1}{2}(\tau_{xy}|_x + \tau_{xy}|_{x+\Delta x} - \tau_{yx}|_y - \tau_{yx}|_{y+\Delta y})\Delta x \Delta y \Delta z$$

or, since $r_g = (\Delta x \Delta y/6)^{\frac{1}{2}}$: radius of gyration,

$$\frac{\rho}{6}\frac{d\Omega}{dt}\Delta x \Delta y = \frac{1}{2}(\tau_{xy}|_x + \tau_{xy}|_{x+\Delta x} - \tau_{yx}|_y - \tau_{yx}|_{y+\Delta y})$$

In the limit as $\Delta x, \Delta y \rightarrow 0$, $\tau_{xy} = \tau_{yx}$.

7.4 Newtonian Fluid

Constitutive equation:

We need a constitutive relation between the stress and the velocity field. Especially, Newtonian fluids.

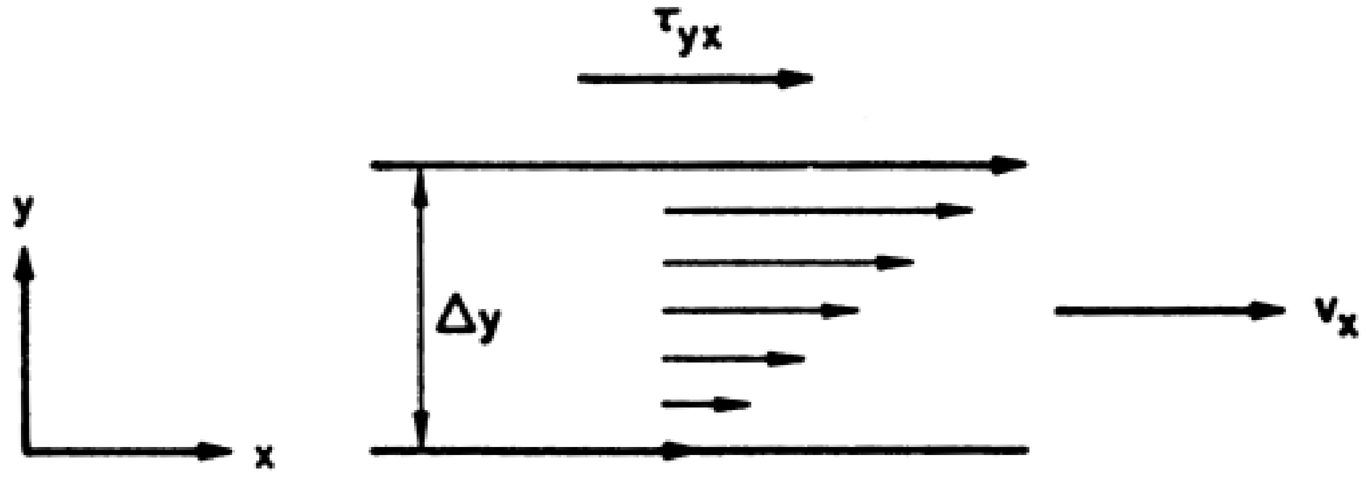

Fig. 7-6. Shearing between two planes of fluid.

Shear stresses:

shear stress $\propto$ shear rate

$\tau_s = \eta \Gamma_s$ where $\tau_s = F/A$, $\Gamma_s = U/H$

x-direction motion only : $\tau_{yx} = \eta \dfrac{d\upsilon_x}{dy} = \tau_{xy}$

y-direction motion only : $\tau_{xy} = \eta \dfrac{d\upsilon_y}{dx} = \tau_{yx}$

Generalization : $\tau_{xy} = \tau_{yx} = \eta \left(\dfrac{\partial \upsilon_x}{\partial y} + \dfrac{\partial \upsilon_y}{\partial x} \right)$

$$\tau_{xz} = \tau_{zx} = \eta \left(\frac{\partial \upsilon_x}{\partial z} + \frac{\partial \upsilon_z}{\partial x} \right)$$

$$\tau_{yz} = \tau_{zy} = \eta \left(\frac{\partial \upsilon_y}{\partial z} + \frac{\partial \upsilon_z}{\partial y} \right)$$

Normal stresses:

$$\sigma_{xx} = -p + 2\eta \frac{\partial \upsilon_x}{\partial x} + (\kappa - \frac{2}{3}\eta) \nabla \cdot \boldsymbol{\upsilon}$$

$$\sigma_{yy} = -p + 2\eta \frac{\partial \upsilon_y}{\partial y} + (\kappa - \frac{2}{3}\eta) \nabla \cdot \boldsymbol{\upsilon}$$

$$\sigma_{zz} = -p + 2\eta \frac{\partial \upsilon_z}{\partial z} + (\kappa - \frac{2}{3}\eta) \nabla \cdot \boldsymbol{\upsilon}$$

κ : bulk viscosity, 0 for monatomic gases
nearly 0 in most cases

Then, we have

$$\sigma_{xx} = -p + 2\eta \frac{\partial \upsilon_x}{\partial x} - \frac{2}{3}\eta \nabla \cdot \boldsymbol{\upsilon}$$

$$\sigma_{yy} = -p + 2\eta \frac{\partial \upsilon_y}{\partial y} - \frac{2}{3}\eta \nabla \cdot \boldsymbol{\upsilon}$$

$$\sigma_{zz} = -p + 2\eta \frac{\partial \upsilon_z}{\partial z} - \frac{2}{3}\eta \nabla \cdot \boldsymbol{\upsilon}$$

And

$$\sigma_{xx}+\sigma_{yy}+\sigma_{zz} = -3p+2\eta\left(\frac{\partial \upsilon_x}{\partial x}+\frac{\partial \upsilon_y}{\partial y}+\frac{\partial \upsilon_z}{\partial z}\right)-2\eta\nabla \cdot \boldsymbol{\upsilon}$$

$$\therefore\ p = -\frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})$$

: a compressive stress that is equal to the mean normal stress on the control volume

Deviatoric stress (or extra stress):

$$\tau_{xx} = \sigma_{xx}+p = 2\eta\frac{\partial \upsilon_x}{\partial x}-\frac{2}{3}\eta\nabla \cdot \boldsymbol{\upsilon}$$

$$\tau_{yy} = \sigma_{yy}+p = 2\eta\frac{\partial \upsilon_y}{\partial y}-\frac{2}{3}\eta\nabla \cdot \boldsymbol{\upsilon}$$

$$\tau_{zz} = \sigma_{zz}+p = 2\eta\frac{\partial \upsilon_z}{\partial z}-\frac{2}{3}\eta\nabla \cdot \boldsymbol{\upsilon}$$

note that $\tau_{xx}+\tau_{yy}+\tau_{zz} = 0$

Newtonian fluids:

* The stress is symmetric.
* The stress at a point in the fluid depends only on the instantaneous value of the velocity gradient at the point.
* The stress is a linear function of the velocity gradients.
* The stress is isotropic when there is no motion.

Pressure:

If the fluid is incompressible,
the thermodynamic pressure is undefined.
For the Newtonian fluid, even when incompressible, it is best to use an isotropic stress given by $-(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})/3$.

Momentum eq'n:

Using the definition $\tau_{xx} = \sigma_{xx} + p$, x component of the Cauchy momenyum eq'n is written as

$$\rho \frac{Dv_x}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x$$

or in terms of the velocity gradients,

$$\rho \frac{Dv_x}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}[2\eta \frac{\partial v_x}{\partial x} - \frac{2}{3}\eta \nabla \cdot \boldsymbol{v}]$$
$$+ \frac{\partial}{\partial y}[\eta(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x})] + \frac{\partial}{\partial z}[\eta(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x})] + \rho g_x$$

y and z components:

$$\rho \frac{Dv_y}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}[2\eta \frac{\partial v_y}{\partial y} - \frac{2}{3}\eta \nabla \cdot \boldsymbol{v}] + \frac{\partial}{\partial z}[\eta(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y})] + \frac{\partial}{\partial x}[\eta(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y})] + \rho g_y$$

$$\rho \frac{Dv_z}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial z}[2\eta \frac{\partial v_z}{\partial z} - \frac{2}{3}\eta \nabla \cdot \boldsymbol{v}] + \frac{\partial}{\partial x}[\eta(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z})] + \frac{\partial}{\partial y}[\eta(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z})] + \rho g_z$$

Navier-Stokes eq'ns:

In many applications the viscosity can be taken as a constant independent of spatial position. Then, we have

$$\rho\frac{Dv_x}{Dt} = -\frac{\partial p}{\partial x}+\eta\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\frac{1}{3}\eta\frac{\partial}{\partial x}(\nabla\cdot\boldsymbol{v})+\rho g_x$$

$$\rho\frac{Dv_y}{Dt} = -\frac{\partial p}{\partial y}+\eta\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\frac{1}{3}\eta\frac{\partial}{\partial y}(\nabla\cdot\boldsymbol{v})+\rho g_y$$

$$\rho\frac{Dv_z}{Dt} = -\frac{\partial p}{\partial z}+\eta\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\frac{1}{3}\eta\frac{\partial}{\partial z}(\nabla\cdot\boldsymbol{v})+\rho g_z$$

: Navier-Stokes Equations

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla p+\eta\nabla^2\boldsymbol{v}+\frac{1}{3}\eta\nabla(\nabla\cdot\boldsymbol{v})+\rho\boldsymbol{g}$$ in vector form

Equivalent pressure:

$$P = p + \rho g h - \frac{1}{3}\eta \nabla \cdot \boldsymbol{v}$$: equivalent pressure

Then, the Navier-Stokes eq'n becomes

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\nabla P + \eta \nabla^2 \boldsymbol{v}$$

7.5 Curvilinear Coordinates

Rectangular (x, y, z)

Cylindrical (r, θ, z)

Spherical (r, θ, ϕ)

7.6 Boundary Conditions

4 differential equations : 1 continuity
3 component eq'ns of Navier-Stokes eq'n

4 variables : 3 components of the velocity
1 pressure

Boundary conditions are required to determine the integration constants.

* No-slip condition along the solid surface.
* Continuity of the velocity and the tangential stress along the fluid-fluid interface. (cf. surface tension effect in the normal stress)
* Symmetry condition.
* Finiteness of velocity or stress.

7.7 Macroscopic Equations

Macroscopic balance equations $\Leftarrow$ microscopic equations

inner product with $\boldsymbol{v}$,
integration over the control volume,
using the Green's theorem
(or Gauss' Divergence theorem)

Assumptions:

* Rectangular conduit with no bend
* Steady state
* Incompressible Newtonian
* Isothermal
* No shaft work

Viscous dissipation term : Rayleigh dissipation function,

$$\Phi = \frac{1}{2}\,\mathrm{II} \quad \text{(Table 8-1)}$$