6. Applications of Macroscopic Balances

6.1 Introduction

Given the flow geometry and the flow rate, asked to calculate the pressure drop, viscous losses, and force.

Generally the number of equations is not sufficient to determine the unknown variables.

Class Ⅰ : Calculate viscous losses for a given flow

some prior information on $\boldsymbol{F}$ ⇒ Δp from the momentum eq'n ⇒ l_V from the energy eq'n

Class Ⅱ : Calculate the force for a given flow (inverse of Class Ⅰ

Class Ⅲ : Calculate the pressure change for a given flow

The Steady Flow of Incompressible Newtonian Fluids

The continuity eq'n :

$$\langle V \rangle_1 A_1 = \langle V \rangle_2 A_2$$

The energy eq'n (Engineering Bernouilli eq'n) :

$$\frac{\alpha_2}{2}\langle V \rangle_2^2 + gh_2 = \frac{\alpha_1}{2}\langle V \rangle_1^2 + gh_1 - \frac{p_2 - p_1}{\rho} + \delta W_S - l_V$$

The Momentum eq'n :

$$\mathbf{0} = w(\beta_1 \langle \boldsymbol{v} \rangle_1 - \beta_2 \langle \boldsymbol{v} \rangle_2) + p_1 \boldsymbol{A}_1 - p_2 \boldsymbol{A}_2 - \boldsymbol{F} + \left(\int_{z_1}^{z_2} \rho A dz\right)\boldsymbol{g}$$

6.2 Losses in Expansion

Let's consider the flow of an incompressible fluid through an expansion

Fig. 6-1. Schematic of flow through an expansion.

Fig. 6-2. Forces on an expansion.

Class I problem : Δp from the momentum equation,

then l_V from the energy equation.

* Assume that the flow is turbulent and that the velocity at points 1 and 2 is uniform over the cross section.

  $\Rightarrow \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1.$ and $\langle V \rangle = V$

* The continuity equation : $A_1V_1 = A_2V_2$
* The momentum equation in the flow direction :

  $$0 = \rho A_1 V_1^2 - \rho A_2 V_2^2 + p_1 A_1 - p_2 A_2 - F$$

  F : the component of $\boldsymbol{F}$ in the flow direction

  Two contributions to F: F_t and F_n

  F_t : the force resulted from the tangential frictional drag along the walls

  F_n : the force resulted from the fluid pressure on the expansion surface at plane e, $= -p_e(A_2 - A_1)$

The momentum equation is then

$$0 = \rho A_1 V_1^2 - \rho A_2 V_2^2 + p_1 A_1 - p_2 A_2 + p_e (A_2 - A_1) - F_t$$

or $$0 = \rho A_1 V_1^2 - \rho A_2 V_2^2 + (p_1 - p_e) A_1 + (p_e - p_2) A_2 - F_t$$

The length of straight pipe is small, $\Rightarrow F_t \approx 0$

$p_e \approx p_1$

$$\therefore \frac{p_1 - p_2}{\rho} = V_2^2 \left(1 - \frac{A_2}{A_1}\right)$$

* The engineering Bernouilli equation :

$h_1 = h_2$, no shaft work

$$l_V = \frac{p_1 - p_2}{\rho} + \frac{1}{2}(V_1^2 - V_2^2)$$

* Substituting the continuity equation for V_1

and the momentum equation for $(p_1 - p_2)/\rho$,

$$l_V = V_2^2 [1 - \frac{A_2}{A_1} + \frac{1}{2} (\frac{A_2}{A_1})^2 - \frac{1}{2}]$$

or $l_V = \frac{V_2^2}{2} (\frac{A_2}{A_1} - 1)^2$: *Borda-Carnot equation*

6.3 Force on a Reducing Bend

Class Ⅱ problem : prior information on l_V

$\Rightarrow$ Δp from the energy eq'n

$\Rightarrow$ $\boldsymbol{F}$ from the momentum eq'n

Assume incompressible turbulent flow ($\alpha = \beta = 1$, $\langle V \rangle = V$)

and no shaft work ($\delta W_S = 0$)

Then, continuity : $A_1 V_1 = A_2 V_2$

energy : $\frac{1}{2} V_2^2 + \frac{p_2}{\rho} = \frac{1}{2} V_1^2 + \frac{p_1}{\rho} - l_V$

x momentum : $0 = \rho A_1 V_1^2 + p_1 A_1 - F_x$

y momentum : $0 = -\rho A_2 V_2^2 - p_2 A_2 - F_y$

l_V : $\quad l_V = \frac{1}{2} K V_2^2$, $\quad K \approx \frac{3}{4}$ from Table 5-1.

Then, we obtain

$$F_x = A_1 (p_1 + \rho V_1^2)$$

$$F_y = -A_2 (p_1 + \frac{1}{2} \rho V_1^2 [1 + (1-K) (\frac{A_1}{A_2})^2])$$

And $\quad |\boldsymbol{F}| = (F_x^2 + F_y^2)^{1/2}$, $\quad \theta = \arctan(\frac{F_y}{F_x})$

If $\frac{p_1}{\rho} \gg \frac{V_1^2}{2}$, $\quad |\boldsymbol{F}| = p_1 (A_1^2 + A_2^2)^{1/2}$, $\quad \theta = \arctan(-\frac{A_2}{A_1})$

6.4 Jet Ejector

Class Ⅲ problem : ⇒ Δp from the momentum eq'n
or the energy eq'n

Fig. 6-4. Schematic of a jet ejector.

It is difficult to estimate the viscous losses in the energy eq'n which will be substantial due to the chaotic mixing.

The frictional force in the momentum eq′n will be negligible in the relatively short distance from 1 to 2.

Therefore, the momentum eq′n can be used to compute Δp.

$$(p_2 - p_1)A = \rho A \beta_1 \langle V \rangle_1^2 - \rho A V_2^2$$

where $\langle V \rangle_1 = V_2 = \lambda^2 V_j + (1-\lambda^2) V_s$

$$\beta_1 = \frac{\langle V^2 \rangle_1}{\langle V \rangle_1^2} = \frac{\lambda^2 V_j^2 + (1-\lambda^2) V_s^2}{\langle V \rangle_1^2}$$

$$\therefore \quad p_2 - p_1 = \lambda^2 (1-\lambda^2) \rho (V_j - V_s)^2$$

6.5 Flow through an Orifice

Class Ⅲ problem : ⇒ Δp from the momentum eq'n

or the energy eq'n

Fig. 6-5. Schematic of flow through an orifice.

The continuity: $AV_1 = A_0 V_2$

The engineering Bernoulli eq′n:

- horizontal flow, no shaft work, $\alpha_1 = \alpha_2 = 1$

$$\frac{1}{2}V_2^2 = \frac{1}{2}V_1^2 + \frac{p_1 - p_2}{\rho} - l_V$$

or

$$\frac{p_1 - p_2}{\rho} = \frac{1}{2}V_2^2\left[1 - \left(\frac{A_0}{A}\right)\right] + l_V$$

$$= \frac{1}{2}V_2^2\left[1 + K - \left(\frac{A_0}{A}\right)^2\right]$$

since $l_V \approx \frac{1}{2}V_2^2 K$

from which $Q = A_0 V_2 = A_0 \sqrt{\frac{1}{1 + K - (A_0/A)^2}}\sqrt{\frac{2(p_1 - p_2)}{\rho}}$

where $K = 1.6[1-(\frac{A_0}{A})^2]$, $\frac{D_0 V_2 \rho}{\eta} > 3x10^4$

(from Perry's Handbook)

Then, we finally have

$$Q = 0.62 A_0 \sqrt{\frac{2(p_1 - p_2)}{[1-(A_0/A)^2]\rho}}$$

orifice coefficient

6.6 Pitot Tube

Fig. 6-6. Schematic of a pitot-static tube.

Class Ⅲ problem : ⇒ Δp from the Bernoulli eq'n

along a streamline

$$\frac{1}{2}V_1^2 + gh_1 = \frac{1}{2}V_2^2 + gh_2 + \int_{p_1}^{p_2} \frac{dp}{\rho} + l_V$$

Along the streamline A from 1 to 2, $V_2 = 0$ and $l_V \approx 0$

$$\frac{1}{2} V_1^2 + \frac{p_1 - p_2}{\rho} = 0 \quad , \qquad p_2 : \text{stagnation pressure}$$

Along the streamline B from 1 to 3, $V_1 = V_2$ and $l_V \approx 0$

$$\therefore \; p_1 = p_2$$

Thus, $$V_1 = \sqrt{\frac{2(p_2 - p_3)}{\rho}}$$

- Accurate within a few percent for the velocity at high Re.
- A correction is required for the neglected l_V at low Re.

6.7 Diameter of a Free Jet

Class Ⅲ problem : ⇒ the flow geometry from the Bernoulli eq'n
or from the momentum eq'n

Fig. 6-10. Schematic of a free jet.

Assume that the parabolic flow persists right up to the tube exit.
(valid for Re greater than 50 to 100)

The continuity eq'n : $$\frac{\pi D^2}{4} \langle V \rangle_1 = \frac{\pi D_j^2}{4} \langle V \rangle_2$$

The Bernoulli eq'n : $p_1 = p_2$, $\alpha_1 = 2$, $\alpha_2 = 1$, and neglecting l_V,

$$\frac{1}{2} < V >_2^2 = < V >_1^2$$

Therefore, $D_j = (\frac{1}{2})^{1/4} D = 0.84D$

The momentum eq'n : $p_1 = p_2 = p_{\text{atm}}$, $\beta_1 = \frac{4}{3}$, $\beta_2 = 1$,

$$w(\frac{4}{3} < V >_1 - < V >_2) + p_{\text{atm}} \frac{\pi}{4}(D^2 - D_j^2) - F = 0$$

Two contributions to F

- The frictional air drag is negligible
- The horizontal component of the force of p_{atm} on the side surface of the jet

$$F = p_{\text{atm}} \frac{\pi}{4}(D^2 - D_j^2)$$

Therefore, the momentum eq′n becomes $\frac{4}{3} < V >_1 = < V >_2$

And we finally have $D_j = (\frac{3}{4})^{1/2} D = 0.866D$

At low Re, substantial velocity rearrangement takes place in the tube exit region. In fact, the jet diameter is 10 to 15% larger than the tube diameter.

* Viscoelastic extrudate swell phenomena

6.8 The Rotameter

Fig. 6-11. Schematic of a rotameter.

Eliminate the pressure difference between the energy and momentum eq′ns ⇒ a single eq′n which expresses the velocity (or flow rate) in terms of geometric and fluid parameters and the losses and forces.
We need some assumptions on the nature of losses and forces.

The taper is gradual. $\Rightarrow$ A_T : constant

The mean velocity : $\langle V \rangle_1 = \langle V \rangle_2 = \dfrac{Q}{A_T}$

Assume that the velocity is uniform over each surface (1 and 2).

$$\alpha_1 = \beta_1 = 1$$

$$\beta_2 = \frac{\langle V^2 \rangle_2}{\langle V \rangle_2^2} = \frac{\left(\dfrac{Q}{A_T - A_B}\right)^2 \left(\dfrac{A_T - A_B}{A_T}\right)}{(Q/A_T)^2} = \frac{A_T}{A_T - A_B}$$

$$\alpha_2 = \frac{\langle V^3 \rangle_2}{\langle V \rangle_2^3} = \left(\frac{A_T}{A_T - A_B}\right)^2$$

With no shaft work, the Bernoulli eq'n is

$$\frac{1}{2}[\alpha_1 \langle V \rangle_1^2 - \alpha_2 \langle V \rangle_2^2] + \frac{p_1 - p_2}{\rho} + g(h_1 - h_2) - l_V = 0$$

Assume that $l_V = \dfrac{1}{2} K_R \left(\dfrac{Q}{A_T - A_B}\right)^2$

The energy eq'n is then

$$\frac{1}{2}\rho\left(\frac{Q}{A_T}\right)^2\left[1-(1+K_R)\left(\frac{A_T}{A_T-A_B}\right)^2\right]+(p_1-p_2)+\rho g(h_1-h_2)=0$$

The momentum eq'n is

$$\rho Q[\beta_1\langle V\rangle_1-\beta_2\langle V\rangle_2]+(p_1-p_2)A_T-\rho g[(h_2-h_1)A_T-V_B]-\rho_B g V_B=0$$

where ρ_B and V_B are the density and volume of the bob.

Substituting for β_2 and $\langle V\rangle$,

$$\rho\left(\frac{Q}{A_T}\right)^2\left(1-\frac{A_T}{A_T-A_B}\right)+(p_1-p_2)+\rho g(h_1-h_2)-\frac{gV_B}{A_T}(\rho_B-\rho)=0$$

Eliminating the pressure difference between the Bernoulli and momentum eq'ns,

$$\frac{1}{2}\left(\frac{Q}{A_T}\right)^2\left(\frac{A_B}{A_T-A_B}\right)^2\left[1+K_R\left(\frac{A_T}{A_B}\right)^2\right]-\frac{gV_B}{A_T}\left(\frac{\rho_B}{\rho}-1\right)=0$$

or $$Q=(A_T-A_B)\sqrt{\frac{A_T/A_B}{1+K_R(A_T/A_B)^2}}\sqrt{\frac{2gV_B}{A_B}\left(\frac{\rho_B}{\rho}-1\right)}$$

K_R : loss coefficient determined experimentally

* The analysis of Schoenborn and Colburn :

$$Q=C_R(A_T-A_B)\sqrt{\frac{2gV_B}{A_B}\left(\frac{\rho_B}{\rho}-1\right)}$$

C_R : rotameter coefficient

K_R and C_R correlate well with Re. (Figs. 6-12, 6-13)

6.9 Flow and Pressure Distribution in a Manifold

Manifold : A device for distributing a liquid or a gas.
The fluid is conveyed through a main tube and ejected through a series of side ports.

- Distribution manifold
- Return manifold

Fig. 6-14. Schematic of a manifold.

Pressure recovery:

The change in pressure distribution ? ⇒ Calss Ⅲ problem

Using the momentum eq′n :

Fig. 6-15. Flow past a single port.

- Neglect the small frictional force on the tube walls.
- The flow V_e is perpendicular to the flow direction.
- Horizontal pipe

$$p_2 - p_1 = \rho (V_1^2 - V_2^2)$$

- Since $V_2 < V_1$, $p_2 > p_1$: pressure increase
- Overestimation of the pressure recovery due to the assumption that The flow V_e is perpendicular to the flow direction.

Using the energy eq′n :

Fig. 6-16. Assumed streamline pattern near a port.

- Neglect the viscous losses

$$p_2 - p_1 = \frac{\rho}{2}(V_1^2 - V_2^2)$$

one-half that calculated from the mementum eq'n

We generally use the following relation

$$p_2 - p_1 = k\rho(V_1^2 - V_2^2) \quad ,$$

k : 0.4 ~ 0.88 (0.45) for distribution manifold

~ 1.0 for return manifold

Side flow:

The conservation of mass :

$$D^2 V_1 = D^2 V_2 + d^2 V_e$$

If the side port is a sharp-edged orifice,

$$V_e = 0.62\sqrt{\frac{2}{\rho}\left(\frac{p_1 + p_2}{2} - p_e\right)}$$

If the side port is a long tube of length l,

$$V_e = \sqrt{\frac{d}{4lf}}\sqrt{\frac{2}{\rho}\left(\frac{p_1+p_2}{2}-p_e\right)}$$

$f \approx 0.005$, $l/d \sim O(10^2)$, then, $\sqrt{\frac{d}{4lf}} \approx 0.5 \sim 1.0$

In general,

$$V_e = c\sqrt{\frac{2}{\rho}\left(\frac{p_1+p_2}{2}-p_e\right)}$$

c : depends on the side port geometry, $O(1)$

Solve the pressure recovery, the conservation of mass, and the side port discharging eq'ns for p_2, V_2, V_e

$$p_2 = \frac{k-\gamma^2}{k+\gamma^2}p_1 + \frac{2\gamma^2}{k+\gamma^2}p_e + \frac{2\rho V_1^2 k^2 \gamma^2}{(k+\gamma^2)^2}\sqrt{1+\frac{2(k+\gamma^2)}{\rho V_1^2 k \gamma^2}(p_1-p_e)}$$

where $\gamma = \dfrac{kcd^2}{D^2} \ll 1$, since kc is less than unity and d^2/D^2 is normally quite small.

Thus, we can expand p_2 about $\gamma = 0$ to obtain

$$p_2 = p_1 + \gamma V_1 \sqrt{8\rho(p_1 - p_e)} + \text{terms of order } \gamma^2$$

$$V_e = c\sqrt{\frac{2}{\rho}[p_1 - p_e + \gamma V_1 (8\rho(p_1 - p_e))^{1/2}]}$$

$$V_2 = V_1 - \frac{d^2}{D^2} V_e$$

6.10 Cavitation

Bernoulli eq'n:

$$p_2 = p_1 + \frac{1}{2}\rho(V_1^2 - V_2^2) + g(h_1 - h_2)$$

If $V_2 >> V_1$, or $h_2 >> h_1$,

$p_2 <$ the vapor pressure of the liquid

$\Rightarrow$ the vapor will form (cavitation)

* near the tip of an impellor

6.11 Compressible Flow

Compressibility is an important factor when the velocity becomes comparable to the velocity of sound.

Sonic velocity can occur in nozzles and in relief valves.

The speed of sound of an isothermal ideal gas is $(p/\rho)^{1/2}$ or $(R_g T/M_w)^{1/2}$.

Pipe flow:

Let's consider the steady-state isothermal flow of an ideal gas in horizontal smooth pipe:

It is found experimentally that the f-Re relation for incompressible fluids applies to compressible fluids as well.

$$\mathrm{Re} = \frac{D<V>\rho}{\eta} = \frac{4}{\pi D}\frac{(\pi D^2<V>\rho/4)}{\eta} = \frac{4w}{\pi D\eta}$$

w : the mass flow rate, independent of axial position

Assume that the viscosity is constant.

Then, Re is constant, and therefore so is the friction factor (f).

The losses:

$$dl_V = 2<V>^2 f\frac{dz}{D}$$

The engineering Bernoulli eq'n for a differential length, assuming that $\alpha = 1$, is

$$\frac{1}{2}\, dV^2 + \frac{dp}{\rho} + 2V^2 f \frac{dz}{D} = 0$$

For an ideal gas, $\rho = \dfrac{M_w p}{R_g T}$

And using the relations, $w = \pi D^2 V \rho / 4$ and $dV^2 = 2V\,dV$

$$-\frac{dp}{p} + \frac{\pi D^2 M_w}{16 w^2 R_g T}\, p\,dp + \frac{2f}{D}\, dz = 0$$

After the integration,

$$-\ln\frac{p_2}{p_1} + \frac{\pi D^2 M_w}{32 w^2 R_g T}(p_2^2 - p_1^2) + \frac{2fL}{D} = 0$$

or $$w^2 = \frac{\pi^2 D^4 \rho_1 p_1}{16}\left[\frac{1-(p_2/p_1)^2}{(4fL/D) - \ln(p_2/p_1)^2}\right]$$

As $p_2 \to p_1$, $w^2 \to 0$. Also $w^2 \to 0$, as $p_2 \to 0$.

Thus, there is a maximum throughput at an intermediate value of p_2.

The maximum is found at

$$\frac{dw^2}{d(p_2/p_1)^2} = \frac{\pi^2 D^4 \rho_1 p_1}{16}\left[-\frac{1}{\frac{4fL}{D} - \ln\left(\frac{p_2}{p_1}\right)^2} + \frac{1-(p_2/p_1)^2}{\left(\frac{p_2}{p_1}\right)\left(\frac{4fL}{D} - \ln\left(\frac{p_2}{p_1}\right)\right)^2}\right] = 0$$

or $$w^2_{max} = \left(\rho_2 \frac{\pi D^2}{4} V_{2,max}\right)^2 = \left(\frac{\pi D^2}{4}\right)^2 p_2^2 \frac{\rho_1}{p_1}$$

and $V_{2,max} = \sqrt{\dfrac{p_2}{\rho_2}}$: the speed of sound at the exit

The exit velocity cannot exceed sonic velocity.

If the pressure outside the pipe
< the pressure at the maximum throughput,
the exit velocity remains at the sonic velocity and
there will be a standing expansion shock wave across which
the pressure changes to the outside value at the exit.

The existence of a maximum throughput is known as choking.