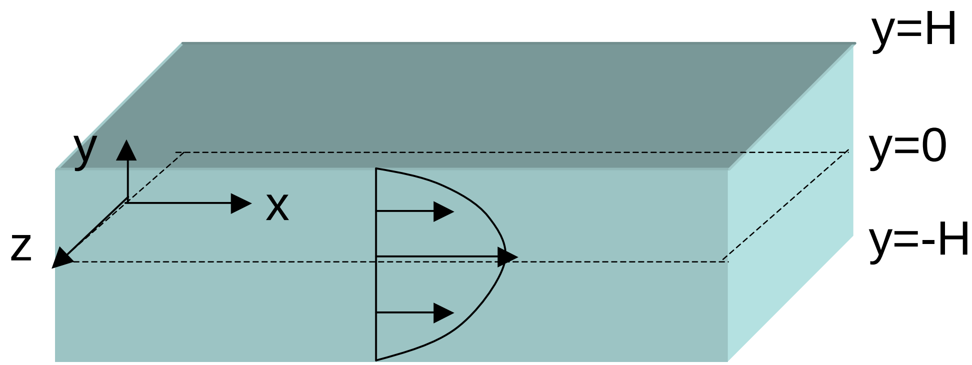
Part III. Linear Stability Analysis of Plane Poiseuille Flow



Equation of Motion: $$\rho \frac{\partial \underline{\tilde{v}}}{\partial \tilde{t}} + \rho \underline{\tilde{v}} \cdot \underline{\tilde{\nabla}} \underline{\tilde{v}} = -\underline{\tilde{\nabla}} \tilde{P} + \mu \underline{\tilde{\nabla}}^2 \underline{\tilde{v}} + \rho \underline{\tilde{f}}$$

Dimensionless variables: $$\underline{v} = \frac{\underline{\tilde{v}}}{u},\ \underline{\nabla} = H\underline{\tilde{\nabla}},\ P = \frac{\tilde{P}}{\rho u^2},\ t = \frac{\tilde{t}}{H/u},\ f = \frac{\tilde{f}H}{u^2}$$

(u: maximum velocity, H: half width)

Dimensionless equation of motion: $$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} = -\underline{\nabla} P + \frac{1}{Re} \underline{\nabla}^2 \underline{v} + \underline{f}$$

$$\left(Re = Hu\rho/\mu \right)$$

(gravity term neglected)

$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}=-\frac{\partial P}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)$$

$$\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}=-\frac{\partial P}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)$$

$$\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}=-\frac{\partial P}{\partial z}+\frac{1}{Re}\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)$$

Dimensionless Continuity Equation: $$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

Boundary Conditions: $v_x=v_y=v_z=0 \quad \text{at} \quad y=\pm 1$

$v_x, v_y, v_z \ \text{bounded at} \ x, z \to \pm\infty$

Consider two-dimensional system with width effect neglected …
By squire’s theorem, 2-D disturbance is the most unstable…

Steady-State: $v_{x,s} = v_{x,s}(y) = 1 - y^2, \;\; v_{y,s} = 0$

$$P_s = P_0 + \left(\frac{\partial P}{\partial x}\right)x$$

Linearized equations:

$$v_x(x,y,t) = v_{x,s}(y) + \hat{v}_x(y)\exp(i\alpha(x-ct))$$

$$v_y(x,y,t) = 0 \qquad + \hat{v}_y(y)\exp(i\alpha(x-ct))$$

$$P(x,y,t) = P_s \qquad + \hat{P}(y)\exp(i\alpha(x-ct))$$

Use streamline function to linearize the system:

(two variables vx, vy can be unified into one variable, ψ)

$$\psi(x,y,t) = \psi_s(y) + \hat{\psi}(y)\exp(i\alpha(x-ct))$$

$$\Rightarrow v_x = -\frac{\partial \psi}{\partial y} = -\frac{\partial \psi_s}{\partial y} - \frac{\partial \hat{\psi}}{\partial y}\exp(i\alpha(x-ct)) = v_{x,s} - \hat{\psi}'\exp(i\alpha(x-ct))$$

$$\Rightarrow v_y = \frac{\partial \psi}{\partial x} = 0 + i\alpha\hat{\psi}\exp(i\alpha(x-ct))$$

Linearized equation of motion (x-direction) :

$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=-\frac{\partial P}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}\right)$$

$$\Rightarrow i\alpha c\hat{\psi}'-i\alpha v_{x,s}\hat{\psi}'+i\alpha v'_{x,s}\hat{\psi}=i\alpha\hat{P}+\frac{1}{Re}\alpha^2\hat{\psi}'-\frac{1}{Re}\hat{\psi}'''$$

Linearized equation of motion (y-direction) :

$$\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}=-\frac{\partial P}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}\right)$$

$$\Rightarrow \alpha^2 c\hat{\psi}-\alpha^2 v_{x,s}\hat{\psi}=-\hat{P}-\frac{1}{Re}i\alpha^3\hat{\psi}+\frac{1}{Re}i\alpha\hat{\psi}''$$

→ Unified equation: (elimination of pressure disturbance):

Orr-Sommerfeld Eqn.

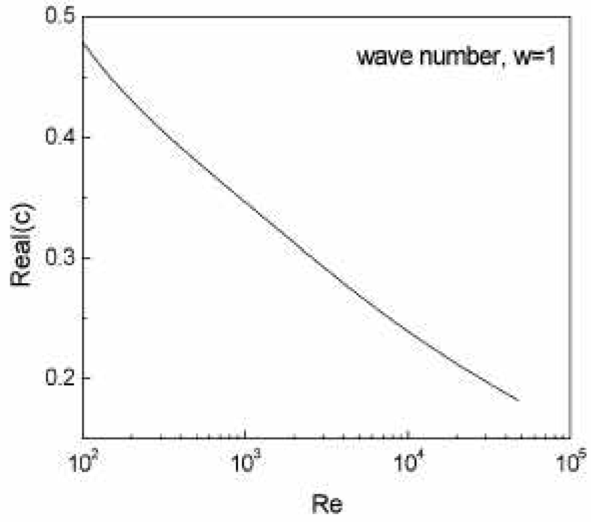
$$i\alpha c\hat{\psi}''-i\alpha^3 c\hat{\psi}=\left(-i\alpha^3 v_{x,s}-\frac{1}{Re}\alpha^4-i\alpha v''_{x,s}\right)\hat{\psi}+\left(\frac{2}{Re}\alpha^2+i\alpha v_{x,s}\right)\hat{\psi}''+\left(-\frac{1}{Re}\right)\hat{\psi}^{iv}$$

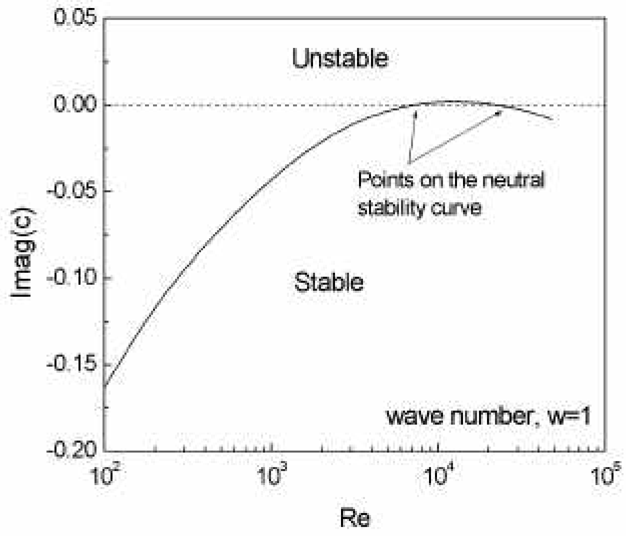
Boundary conditions:

$$\hat{\psi}(0)=\hat{\psi}(1)=0$$

$$\hat{\psi}'(0)=\hat{\psi}'(1)=0$$

Real and imaginary parts of the first eigenvalue of the Orr-Sommerfeld equation for w=1





Neutral stability curve