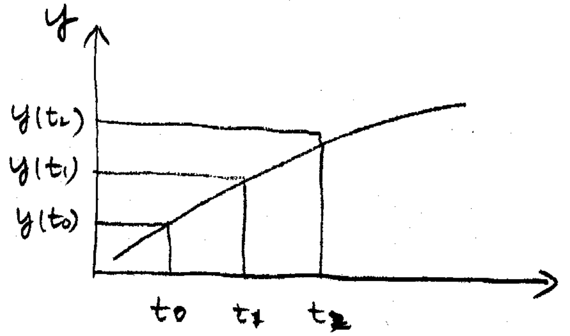
2 장 근사 함수

1. Lagrangian Approximation



Lagrangian Interpolation $\{t_j\}$

→ Construction of a polynomial that passes through the points.

Set $\{t_i\}$ that discretizes time

Construct $\bar{y}(t) \simeq y(t)$

◦ Find $\bar{y}(t) = \sum_{j=0}^{n} c_j t^j$

≡ Find coefficients $\{c_j\}$ which satisfies

→ $\bar{y}(t_i) = y(t_i)$

For $k = 0, 1, \cdots n$

$$\sum_{j=0}^{n} c_j t_k^j = y(t_k)$$

System of linear equations

$$\underline{A}\,\underline{c} = \underline{b}$$

- Lagrange interpolation polynomial

$$f(x) \cong \bar{f}(x) = \sum_{j=0}^{n} f(x_j) L_j(x)$$

$$L_j(x_i) = 0 \quad i \neq j$$
$$L_j(x_i) = 1 \quad i = j$$

For $n=1$,

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}, \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

For n

$$L_j(x) = \frac{(x-x_0)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}$$

$$= \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x - x_i)}{(x_j - x_i)}$$: Lagrange interpolating polynomial

<u>Theorem</u>

If $x_0, x_1 \cdots x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$, then, for each x in $[a, b]$ and a number $\xi(x)$ in (a, b) exists with

$$f(x) = \bar{f}(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

where $\bar{f}(x) = \sum_{j=0}^{n} f(x_j) L_j(x)$

2. Divided Difference

$$\bar{f}(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

a_0 : $\quad a_0 = \bar{f}(x_0) = f(x_0)$

a_1 : $\quad f(x_0) + a_1(x_1 - x_0) = \bar{f}(x_1) = f(x_1)$

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

zero th divided difference of the function f w.r.t x_i. $f[x_i]$

$$f[x_i] = f(x_i)$$

first divided difference of f wrt x_i, x_{i+1}

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

k-th divided difference of f wrt $x_i, x_{i+1} \cdots x_{i+k}$

$$f[x_i, \cdots x_{i+k}] = \frac{f[x_{i+1}, \cdots x_{i+k}] - f[x_i, \cdots x_{i+k-1}]}{x_{i+k} - x_i}$$

$$\bar{f}(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \cdots x_k](x-x_0)\cdots(x-x_{k-1})$$

~ Newton's interpolatory divided-difference formula

For equal spacing, $h = x_{i+1} - x_i$,

$\rightarrow \quad x = x_0 + \alpha h$

$x - x_i = (\alpha - i) h$

$$\bar{f}(x) = \bar{f}(x_0 + \alpha h)$$
$$= f[x_0] + \alpha h \, f[x_0, x_1]$$
$$+ \alpha(\alpha-1) h^2 f[x_0, x_1, x_2]$$
$$+ \cdots + \alpha(\alpha-1)\cdots(\alpha-n+1) h^n f[x_0, x_1, \cdots x_n]$$
$$= \sum_{k=0}^{n} \alpha(\alpha-1)\cdots(\alpha-k+1) h^k f[x_0, x_1 \cdots x_k]$$

$${}_{\alpha}C_k = \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

$$\bar{f}(x) = \bar{f}(x_0 + \alpha h)$$
$$= \sum_{k=0}^{n} \binom{\alpha}{k} k! \, h^k f[x_0, x_1 \cdots x_n]$$

~ Newton forward divided-difference formula

* Newton forward-difference formula

$$\Delta f(x_i) = f(x_{i+1}) - f(x_i)$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h}\Delta f(x_0)$$

$$f[x_0, x_1, x_2] = \frac{1}{2h}\left[\frac{\Delta f(x_1) - \Delta f(x_0)}{h}\right]$$

$$= \frac{1}{2h^2}\Delta^2 f(x_0)$$

$$\vdots$$

$$f[x_0, x_1, \cdots x_k] = \frac{1}{k!\,h^k}\Delta^k f(x_0)$$

$$\bar{f}(x) = \sum_{k=0}^{n} \binom{\alpha}{k} \Delta^k f(x_0)$$

* If the interpolating nodes are reordered as $x_n, x_{n-1} \cdots x_0$

$$\bar{f}(x) = f[x_n] + f[x_{n-1}, x_n](x - x_n)$$
$$+ f[x_{n-2}, x_{n-1}, x_n](x - x_n)(x - x_{n-1})$$
$$+ \cdots + f[x_0, \cdots x_n](x - x_n) \cdots (x - x_1)$$

For equal spacing

$$\bar{f}(x) = \bar{f}(x_n + \alpha h) \qquad \alpha < 0$$
$$= f[x_n] + \alpha h f[x_{n-1}, x_n] + \cdots$$
$$+ \alpha(\alpha+1) \cdots (\alpha+n-1) h^n f[x_0, \cdots x_n]$$

~ Newtons backward divided-difference formula

$$\nabla f(x_i) = f(x_i) - f(x_{i-1})$$

$$f[x_{n-1}, x_n] = \frac{1}{h} \nabla f(x_n)$$

$$f[x_{n-2}, x_{n-1}, x_n] = \frac{1}{2h^2} \nabla^2 f(x_n)$$

$$\vdots$$

$$f[x_{n-k}, \cdots x_{n-1}, x_n] = \frac{1}{k!\,h^k} \nabla^k f(x_n)$$

$$\bar{f}(x) = f[x_n] + \alpha \nabla f(x_n) + \frac{\alpha(\alpha+1)}{2} \nabla^2 f(x_n) + \cdots + \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} \nabla^n f(x_n)$$

$$\binom{-\alpha}{k} = \frac{-\alpha(-\alpha-1)\cdots(-\alpha-k+1)}{k!} = (-1)^k \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}$$

$$\bar{f}(x) = \sum_{k=0}^{n} (-1)^k \binom{-\alpha}{k} \nabla^k f(x_n)$$

Newton backward-difference formula

3. Hermite Interpolation

Lagrange interpolation : $\bar{f}(x_k) = f(x_k)$

Hermite interpolation: $\bar{f}(x_k) = f(x_k)$

$\bar{f}'(x_k) = f'(x_k)$

~ osculating polynomial

$$\bar{f}(x) = \sum_{j=0}^{n} f(x_j) H_j(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_j(x)$$

degree (2n+1) at most

$$\bar{f}(x_k) = f(x_k) \rightarrow H_j(x_k) = \delta_{jk}$$
$$\hat{H}_j(x_k) = 0 \rightarrow$$

$$\bar{f}'(x_k) = f'(x_k) \rightarrow H_j'(x_k) = 0 \rightarrow$$
$$\hat{H}_j'(x_k) = \delta_{jk}$$

$L_j(x_k) = \delta_{jk}$ degree n

$[L_j(x_k)]^2 = \delta_{jk}$ degree $2n$

$\frac{d}{dx}[L_j(x_k)]^2 = 0$ when $i \neq j$

$$H_j(x) = r_j(x)[L_j(x)]^2$$
$$\hat{H}_j(x) = S_j(x)[L_j(x)]^2$$

$r_j(x)$, $S_j(x)$: linear function of x

$$r_j(x_j) = 1, \quad r_j'(x_j) + \cancel{r_j}(x_j) \cdot 2 L_j'(x_j) = 0$$

$$S_j(x_j) = 0 \quad S_j'(x_j) + \cancel{S_j}(x_j) \cdot 2 L_j'(x_j) = 1$$

$[L_j(x_j)]^2 = 1$, $L_j(x_j)$, $= 0$, $[L_j(x_j)]^2$, $L_j(x_j)$

$$r_j(x_j) = 1, \quad r_j'(x_j) + 2L_j'(x_j) = 0$$
$$s_j(x_j) = 0, \quad s_j'(x_j) = 1$$

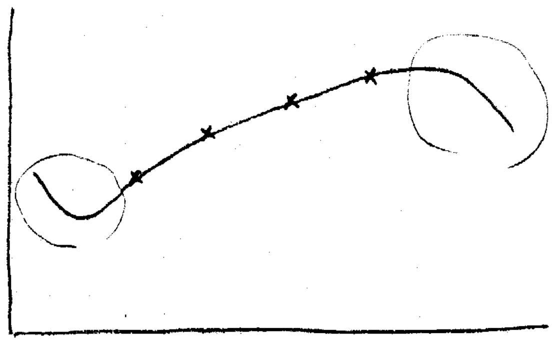
$$\begin{cases} r_j(x) = 1 - 2L_j'(x_j)(x - x_j) \\ s_j(x) = x - x_j \end{cases}$$

$$\bar{f}(x) = \sum_{j=0}^{n} f(x_j) H_j(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_j(x)$$
$$H_j(x) = [1 - 2L_j'(x_j)(x - x_j)] [L_j(x)]^2$$
$$\hat{H}_j(x) = (x - x_j) [L_j(x)]^2$$

~ Hermite interpolating polynomial

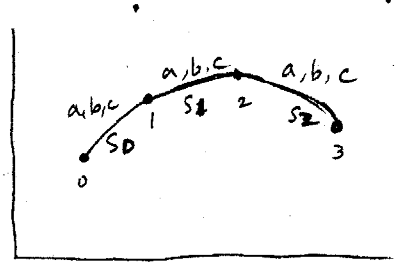
4. Cubic Spline Interpolation

Global interpolation



piecewise polynomial approximation

- piecewise linear interpolation
  - ~ lose differentiability
- piecewise polynomial of Hermite type
  - ~ should know the derivative of function approximated.
- piecewise quadratic polynomial



$$f(x_j) = S(x_j) \qquad j = 0, 1, 2, 3 \qquad 4$$

$$S_j(x_{j+1}) = S_{j+1}(x_{j+1}) \qquad j = 0, 1 \qquad 2$$

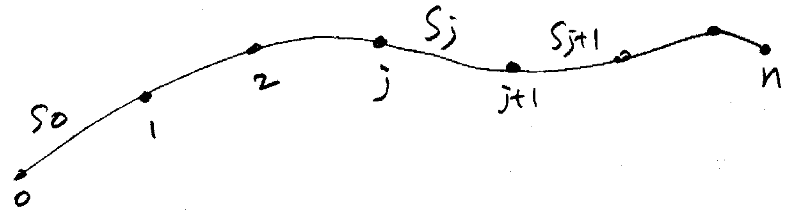
$$S_j'(x_{j+1}) = S_{j+1}'(x_{j+1}) \qquad j = 0, 1 \qquad 2$$

dof : 1

problem when the derivatives at ~~the~~ both end points ~~are~~ should be specified.

- piecewise cubic polynomial

  ~ Cubic spline interpolation



$4(n-1)$

$n+1$

$$S(x_j) = f(x_j) \qquad j = 0, \cdots n$$

$$S_j(x_{j+1}) = S_{j+1}(x_{j+1}) \qquad j = 0, 1 \cdots n-2$$

$$S_j'(x_{j+1}) = S_{j+1}'(x_{j+1}) \qquad j = 0, 1 \cdots n-2$$

$$S_j''(x_{j+1}) = S_{j+1}''(x_{j+1}) \qquad j = 0, 1 \cdots n-2$$

$3(n-1)$

$4n-2$

Free or natural boundary

$$S''(x_0) = S''(x_n) = 0$$

Clamped boundary

$$S'(x_0) = f'(x_0), \quad S'(x_n) = f'(x_n)$$

where S_j is a cubic polynomial defined on the subinterval $[x_j, x_{j+1}]$

4-n

$$S_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3$$

$j = 0, \cdots n-1$

$$S_j(x_j) = a_j = f(x_j)$$ n n

$$a_{j+1} = S_j(x_{j+1}) = S_{j+1}(x_{j+1})$$ $n-1$

$$= a_j + b_j(x_{j+1}-x_j) + c_j(x_{j+1}-x_j)^2 + d_j(x_{j+1}-x_j)^3$$

$h_j \equiv x_{j+1}-x_j$

$$\stackrel{\cdot}{=} a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 \qquad j=0\cdots n-2 \quad (1)$$

$\underline{a_n = f(x_n)}$ 1

$$S_j'(x) = b_j + 2c_j(x-x_j) + 3d_j(x-x_j)^2$$

$$S_j'(x_j) = b_j \qquad j = 0, \cdots n-1$$

$$b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2 \qquad (2) \quad j=0,\cdots n-1 \quad n$$

$\underline{b_n = S'(x_n)}$

$$c_{j+1} = c_j + 3d_j h_j \qquad j=0,\cdots n-1 \quad (3)$$

$\underline{c_n = S''(x_n)/2}$

(3)→(1),(2)

$$a_{j+1} = a_j + b_j h_j + \frac{h_j^2}{3}(2c_j + c_{j+1}) \quad (4)$$

$$b_{j+1} = b_j + h_j(c_j + c_{j+1}) \qquad j=0,1\cdots n-1 \quad (5)$$

$$h_{j-1}c_{j-1} + 2(h_{j-1}+h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1}-a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1})$$

$j = 1, \cdots n-1$

(4)→ $b_j = \frac{a_{j+1}-a_j}{h_j} - \frac{h_j}{3}(2c_j + c_{j+1})$

$\underline{A}\,\underline{x} = \underline{b}$ unknown

$\underline{x} = \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix}$ eqn: $n-1$

i) $S''(x_0) = S''(x_n) = 0.$

$$c_n = S''(x_n)/2 = 0$$

$$S''(x_0) = 0 = 2c_0 + 6d_0(x_0 - x_0)$$

$$= 2c_0 \rightarrow c_0 = 0$$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 0 & 0 & & & & \phi \\ h_0 & 2(h_0+h_1) & h_1 & & & & \\ & h_1 & 2(h_1+h_2) & h_2 & & & \\ & & & \ddots & & & \\ & \phi & & & & & \\ & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix}$$

ii) $S'(x_0) = f'(x_0) = \alpha,\ S'(x_n) = f'(x_n) = \beta$

$$b_0 = \frac{a_1 - a_0}{h_0} - \frac{h_0}{3}(2c_0 + c_1) = \alpha$$

$$2h_0 c_0 + h_0 c_1 = \frac{3}{h_0}(a_1 - a_0) - 3\alpha$$

$$b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n) = \beta$$

$$\frac{a_n - a_{n-1}}{h_{n-1}} - \frac{h_{n-1}}{3}(2c_{n-1} + c_n) + h_{n-1}(c_{n-1} + c_n) =$$

$$= \frac{a_n - a_{n-1}}{h_{n-1}} + \frac{h_{n-1}}{3}(c_{n-1} + 2c_n) = \beta$$

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3\beta - \frac{3}{h_{n-1}}(a_n - a_{n-1})$$

$$\underline{\underline{A}} = \begin{bmatrix} 2h_0 & h_0 & 0 & & & \\ h_0 & 2(h_0 + h_1) & h_1 & & & \\ & h_1 & 2(h_1 + h_2) & h_2 & & \\ & & & \ddots & & \\ & & & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1} \\ & & & & h_{n-1} & 2h_{n-1} \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3\alpha \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3\beta - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{bmatrix}$$