2.8 Nonlinear Equations

General nonlinear equation

$$\underline{R}(\underline{x}) = \underline{0}$$

That is

$$\underline{R} : \underline{x} \in \Re^n \rightarrow \underline{0} \in \Re^n$$

In a scalar notation

$$\begin{aligned} R_1(x_1, x_2, \ldots, x_n) &= 0 \\ R_2(x_1, x_2, \ldots, x_n) &= 0 \\ \vdots \quad &\vdots \quad \vdots \\ R_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

Characteristics :

1. Cannot be usually solved in a finite number of operations.
   $\rightarrow$ require iteration
2. Require an initial guess. Ability of scheme to converge to solution depends on the guess.
3. Only guarantees convergence in a limited number of cases.

Methods :

1. Sequential methods: Fixed set of operations leading to a sequence of $\{\underline{x}^{(k)}\}_{k\rightarrow\infty} \rightarrow \underline{x}^*$.
2. Nonsequential method: Involve random selection

Sequential methods :
Each is characterized by an iteration formula

$$\underline{x}^{(k+1)} = \underline{x}^{(k)} + \lambda^{(k)}\underline{D}^{(k)}$$

where

$\underline{D}^{(k)}$: correction vector
$\lambda^{(k)}$: relaxation parameter

- Trade-off: Work involved in finding $\underline{D}$ vs. accuracy of solution
- Fixed point iteration:
  Let $\lambda = 1$ and $\underline{D} = \underline{R}$, then

  $$\underline{x}^{(k+1)} = \underline{x}^{(k)} + \underline{R}(\underline{x}^{(k)})$$

  This iteration defines a sequence $\{\underline{x}^{(k)}\} \to \underline{x}^*$, where $\underline{x}^*$ is a solution of $\underline{R}(\underline{x}^*) = \underline{0}$ and is a **fixed point**.

Many ways to write a fixed point algorithm.
For example, there are several algorithms for

$$R(x) = x^3 - 7x - 6 = 0$$

1. $$\frac{1}{7}x^3 - x - \frac{6}{7} = 0$$

$$x^{(k+1)} = \frac{{x^{(k)}}^3}{7} - \frac{6}{7}$$

2. $$\frac{x^3 - 7x - 6}{-x^2} = 0$$

$$x^{(k+1)} = \frac{7x^{(k)} + 6}{{x^{(k)}}^2}$$

3. $$\frac{x^3 - 7x - 6}{-(3x^2 - 7)} = 0$$

$$x^{(k+1)} = \frac{2{x^{(k)}}^3 + 6}{3{x^{(k)}}^2 - 7}$$

This algorithm is Newton's method.

Apply three algorithms above for $R(x) = x^3 - 7x - 6 = 0$
and iterate until $|x^{(k+1)} - x^{(k)}| \leq 10^{-5}$.
The exact solution is $x = -1, -2, 3$
from $R(x) = x^3 - 7x - 6 = (x+1)(x+2)(x-3) = 0$.

Result of iteration			
Initial guess	Algorithm		
	(1)	(2)	(3)
$x^{(0)} = -1.1$	$n = 12$ $x^{(12)} = -1.00000$	$n = 10$ $x^{(10)} = -2.00000$	$n = 3$ $x^{(3)} = -1.00000$
$x^{(0)} = -2.2$	$n = 6$ $\lvert x^{(6)}\rvert > 10^6$	$n = 9$ $x^{(9)} = -2.00000$	$n = 4$ $x^{(4)} = -2.00000$

The different behaviour is explained by Contraction Mapping Theorm.

Contraction Mapping Theorem .

1. Let $\underline{\phi}(\underline{x})$ be a continuous set of functions that map a closed and bounded region $\mathcal{R} \in \Re^n$ into itself. When $\underline{x} = (x_1, x_2, \ldots, x_n)^T \in \mathcal{R}$, it follows that

$$\underline{\phi}(\underline{x}) = \begin{bmatrix} \phi_1(x_1, x_2, \ldots, x_n) \\ \phi_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ \phi_n(x_1, x_2, \ldots, x_n) \end{bmatrix} \in \mathcal{R}$$

Example : For $\underline{\phi}(\underline{x}) = \underline{x} + \underline{R}(\underline{x})$ trying to solve $\underline{R}(\underline{x}) = 0$, fixed point algorithm is written as

$$\underline{x}^{(k+1)} = \underline{x}^{(k)} + \underline{R}(\underline{x}^{(k)})$$

Then, the solution satisfies $\underline{x} = \underline{\phi}(\underline{x})$.

2. Assume that there exists a positive constant $L < 1$, such that

$$\|\underline{\phi}(\underline{a}) - \underline{\phi}(\underline{b})\| \leq L\|\underline{a} - \underline{b}\| \ \forall \underline{a}, \underline{b} \in \mathcal{R}$$

Then in $\mathcal{R}$ there is a unique solution of the equation

$$\underline{x} = \underline{\phi}(\underline{x})$$

and the sequence $\{\underline{x}^{(k)}\}$ defined by

$$\underline{x}^{(k+1)} = \underline{\phi}(\underline{x}^{(k)})$$

converges to this solution for any initial approxiamtion $\underline{x}^{(0)} \in \mathcal{R}$. Here L is called a Lipschitz constant.

Note that the contraction mapping theorem is a sufficient but not necessary condition for convergence.

2.9 Iterative Solution of Linear Equations

For $\underline{\underline{A}}\,\underline{x} = \underline{b}$, the residual equation is

$$\underline{R}(\underline{x}) = \underline{\underline{A}}\,\underline{x} - \underline{b} = \underline{0}$$

and

$$\underline{\phi}(\underline{x}) = \underline{\underline{A}}\,\underline{x} - \underline{b} + \underline{x}$$

Then,

$$\begin{aligned} \underline{x} &= \underline{\phi}(\underline{x}) \\ &= \underline{\underline{A}}\,\underline{x} - \underline{b} + \underline{x} \\ &= (\underline{\underline{A}} + \underline{\underline{I}})\underline{x} - \underline{b} \\ &= \underline{\underline{M}}\,\underline{x} + \underline{g} \end{aligned}$$

Fixed Point Iteration is

$$\begin{aligned} \underline{x}^{(k+1)} &= \underline{\phi}(\underline{x}^{(k)}) \\ &= \underline{\underline{M}}\,\underline{x}^{(k)} + \underline{g} \end{aligned}$$

Here $\underline{\underline{M}}$ plays a role of $\underline{\underline{J}}$. The condition for convergence is $L = \|\underline{\underline{M}}\| < 1$.

Splitting of $\underline{\underline{A}}$.

For $\underline{\underline{A}}\,\underline{x} = \underline{b}$, $\underline{\underline{A}}$ is splitted as

$$\underline{\underline{A}} = \underline{\underline{B}} - \underline{\underline{C}}$$

where $\underline{\underline{B}}$ is non-singular. Then,

$$\underline{\underline{B}}\,\underline{x} - \underline{\underline{C}}\,\underline{x} = \underline{b}$$

and

$$\underline{x} = \underbrace{\underline{\underline{B}}^{-1}\underline{\underline{C}}}_{\underline{\underline{M}}}\,\underline{x} + \underbrace{\underline{\underline{B}}^{-1}\underline{b}}_{\underline{g}}$$

1. Jacobi Method

$$\underline{\underline{A}} = \underline{\underline{D}} + \underline{\underline{L}} + \underline{\underline{U}}$$

$$\begin{aligned}\underline{\underline{D}} &= \text{diagonal element of } \underline{\underline{A}} \\ &= \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \phi \\ & & \ddots & \\ & \phi & & a_{nn} \end{bmatrix}\end{aligned}$$

$$L_{ij} = \begin{cases} a_{ij} & i > j \\ 0 & i \le j \end{cases} \quad U_{ij} = \begin{cases} a_{ij} & i < j \\ 0 & i \ge j \end{cases}$$

$$\begin{aligned}(\underline{\underline{D}} + \underline{\underline{L}} + \underline{\underline{U}})\underline{x} &= \underline{b} \\ \underline{\underline{D}}\underline{x} &= -(\underline{\underline{L}} + \underline{\underline{U}})\underline{x} + \underline{b} \\ \underline{x} &= -\underline{\underline{D}}^{-1}(\underline{\underline{L}} + \underline{\underline{U}})\underline{x} + \underline{\underline{D}}^{-1}\underline{b}\end{aligned}$$

$$x_i^{(k+1)} = \frac{-\sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij} x_j^{(k)} + b_i}{a_{ii}}, \quad i = 1, 2, \ldots, N$$

2. Gauss-Seidel Method

$$\begin{aligned}(\underline{\underline{D}} + \underline{\underline{L}})\underline{x}^{(k+1)} &= -\underline{\underline{U}}\,\underline{x}^{(k)} = \underline{b} \\ \underline{\underline{D}}\,\underline{x}^{(k+1)} &= -\underline{\underline{L}}\,\underline{x}^{(k+1)} - \underline{\underline{U}}\,\underline{x}^{(k)} + \underline{b}\end{aligned}$$

$$x_i^{(k+1)} = \frac{-\sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} + -\sum_{j=i+1}^{N} a_{ij} x_j^{(k)} + b_i}{a_{ii}}, \quad i = 1, 2, \ldots, N$$

Sequentially updates as information is available.

When both converges, GS needs fewer iteration than Jacobi.

3. Successive Over-relaxation (SOR)

$$\begin{aligned}
(\underline{\underline{D}} + \underline{\underline{L}} + \underline{\underline{U}})\underline{x} &= \underline{b} \\
\underline{\underline{D}}\,\underline{x} &= \underline{\underline{D}}\,\underline{x} + \omega(-\underline{\underline{D}} - \underline{\underline{L}} - \underline{\underline{U}})\underline{x} + \omega\underline{b} \\
(\underline{\underline{D}} + \omega\underline{\underline{L}})\underline{x} &= (1-\omega)\underline{\underline{D}}\,\underline{x} - \omega\underline{\underline{U}}\,\underline{x} + \omega\underline{b}
\end{aligned}$$

$$\begin{aligned}
(\underline{\underline{D}} + \omega\underline{\underline{L}})\underline{x}^{(k+1)} &= (1-\omega)\underline{\underline{D}}\,\underline{x}^{(k)} - \omega\underline{\underline{U}}\,\underline{x}^{(k)} + \omega\underline{b} \\
\underline{\underline{D}}\underline{x}^{(k+1)} &= (1-\omega)\underline{\underline{D}}\,\underline{x}^{(k)} - \omega\underline{\underline{L}}\,\underline{x}^{(k+1)} - \omega\underline{\underline{U}}\,\underline{x}^{(k)} + \omega\underline{b}
\end{aligned}$$

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(-\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{N} a_{ij}x_j^{(k)} + b_i\right)$$

Rate of convergence :

For 1-D problem

$$x^{(k+1)} = \phi(x^{(k)}) = x^{(k)} + R(x^{(k)}) \tag{2.5}$$

Let

$$x^* = \phi(x^*) \tag{2.6}$$

Subtracting Eq. (2.5) from Eq. (2.6) gives

$$x^* - x^{(k+1)} = \phi(x^*) - \phi(x^{(k)})$$

Then,

$$\begin{aligned}
\|x^* - x^{(k+1)}\| &= \|\phi(x^*) - \phi(x^{(k)})\| \\
&\leq L\|x^* - x^{(k)}\|
\end{aligned}$$

where $L = \max|\phi'(\xi)|$. And,

$$\epsilon^{(k+1)} \leq L\epsilon^{(k)}$$

shows linear convergence.

Order of convergence :

When

$$\lim_{k\to\infty} \frac{|\epsilon^{(k+1)}|}{|\epsilon^{(k)}|^p} = \text{constant}$$

p is an order of convergence.

- Linear convergence

$$\|\epsilon^{(k+1)}\| \leq L\|\epsilon^{(k)}\|$$

where $\epsilon^{(k)} = x^* - x^{(k)}$.

- Higher order convergence

$$\|\epsilon^{(k+1)}\| \leq L\|\epsilon^{(k)}\|^p$$

Taylor series about the exact solution

$$x^* = \phi(x^*) \tag{2.7}$$

$$\begin{aligned} x^{(k+1)} &= \phi(x^{(k)}) \\ &\cong \phi(x^*) + \phi'(x^*)(x^{(k)} - x^*) + \frac{\phi''(x^*)}{2}(x^{(k)} - x^*)^2 + \cdots \end{aligned} \tag{2.8}$$

Subtracting Eq. (2.7) from Eq. (2.8) gives

$$x^{(k+1)} - x^* = \phi'(x^*)(x^{(k)} - x^*) + \frac{\phi''(x^*)}{2}(x^{(k)} - x^*)^2 + \cdots$$

Then,

$$\lim_{x^{(k)}\to x^*} \frac{x^{(k+1)} - x^*}{x^{(k)} - x^*} = \phi'(x^*) \text{ when } \phi'(x^*) \neq 0$$

This is linear convergence.

As a special case, when $\phi'(x^*) = 0$,

$$x^{(k+1)} - x^* = \frac{\phi''(x^*)}{2}(x^{(k)} - x^*)^2 + \cdots$$

And,

$$\lim_{x^{(k)}\to x^*} \frac{x^{(k+1)} - x^*}{(x^{(k)} - x^*)^2} = \phi''(x^*)$$

This corresponds to quadratic convergence.

Newton's method :

In 1-D case, we want to solve $R(x) = 0$. During the iteration

$$\begin{aligned} R(x^{(k+1)}) &= 0 \\ &= R(x^{(k)}) + R'(x^{(k+1)} - x^{(k)}) + \mathcal{O}[(x^{(k+1)} - x^{(k)})^2] \end{aligned}$$

Then,

$$x^{(k+1)} = x^{(k)} - \frac{R(x^{(k)})}{R'(x^{(k)})}$$

In this case,

$$\phi(x) = x - \frac{R(x)}{R'(x)}$$

and

$$\phi'(x) = \frac{R''R}{(R')^2}$$

For $x = x^*$, $R(x^*) = 0$ and $\phi'(x^*) = 0$ if $R' \neq 0$.

Thus, Newton's method shows quadratic convergence.

In muliple dimension

$$\begin{aligned} \underline{R}(\underline{x}^{(k+1)}) &= \underline{0} \\ &= \underline{R}(\underline{x}^{(k)}) + \left[\frac{\partial \underline{R}}{\partial \underline{x}}\right]_{\underline{x}^{(k)}} (\underline{x}^{(k+1)} - \underline{x}^{(k)}) \end{aligned}$$

Here,

$$\left[\frac{\partial \underline{R}}{\partial \underline{x}}\right]_{\underline{x}^{(k)}} = \underline{\underline{J}}(\underline{x}^{(k)})$$

called Jacobian matrix. Then,

$$\begin{aligned} \underline{x}^{(k+1)} - \underline{x}^{(k)} &= \underline{\delta}^{(k+1)} \\ &= -\underline{\underline{J}}^{-1}(\underline{x}^{(k)})\underline{R}(\underline{x}^{(k)}) \end{aligned}$$

and $\underline{x}$ is updated by

$$\underline{x}^{(k+1)} = \underline{x}^{(k)} + \underline{\delta}^{(k+1)}$$

Simple Newton iteration :

In this method Jacobian matrix is not updated.

Adaptive Newton method :

- Full Newton: Update $\underline{\underline{J}}$ at each iteration
- Simple Newton: Never updates $\underline{\underline{J}}$
- Adaptive Newton: Update $\underline{\underline{J}}$ depending on the rate of convergence