5장 편미분 방정식

* Elliptic Problems in 2 space dimensions

$$\nabla^2 C + R(C) = 0$$

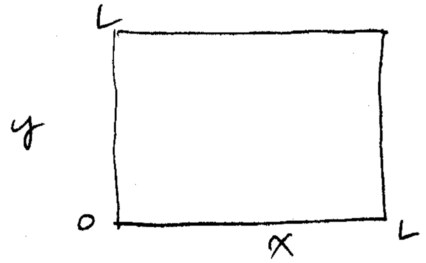
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

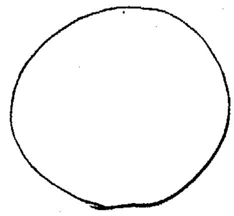
* Evolution Problem (Parabolic Problem)

$$\frac{\partial C}{\partial t} = \nabla^2 C + R(C)$$

1. Elliptic Problems

* Types of Domain



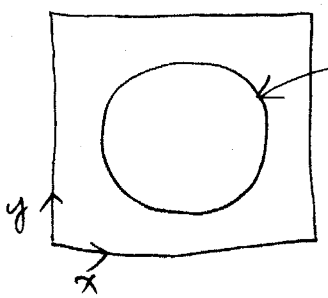


$0 \le r \le R$

$0 \le \theta \le 2\pi$

"Regular Domains"

Boundary is coordinate line

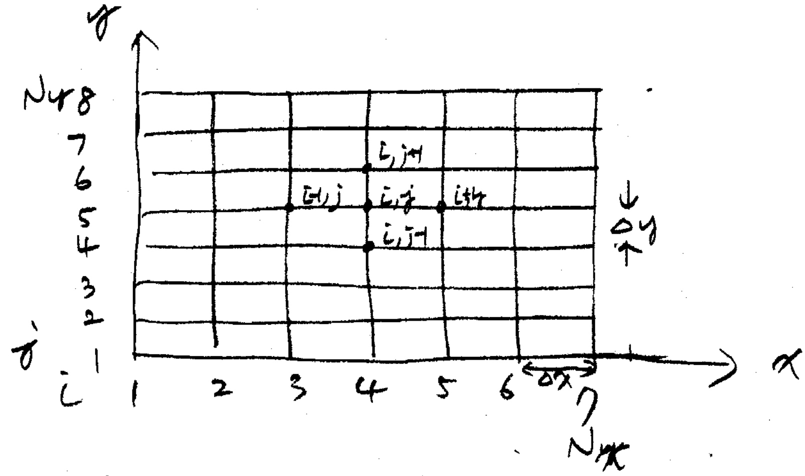


Boundary is not a coordinate line

"irregular domain"

(1) Finite Difference Methods for Regular Domain

$\nabla^2 c = 0$ on a square (c is given on boundary)



$c(x_i, y_j) = c_{ij}$

$$\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0$$

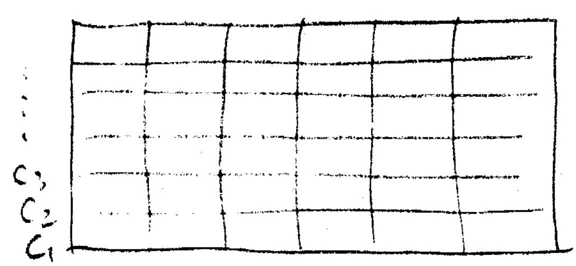
Apply central Difference

$$\frac{c_{i+1,j} - 2c_{ij} + c_{i-1,j}}{\Delta x^2} + \frac{c_{i,j+1} - 2c_{ij} + c_{i,j-1}}{\Delta y^2} = 0$$

$i = 2 \cdots N_x - 1$, $j = 2 \cdots N_y - 1$

Define $k = j + (i-1) * N_y$ (A choice!) can change to $i + (j-1) * N_x$

$\underline{c}^T = (c_1, c_2 \cdots c_{N_x * N_y})$

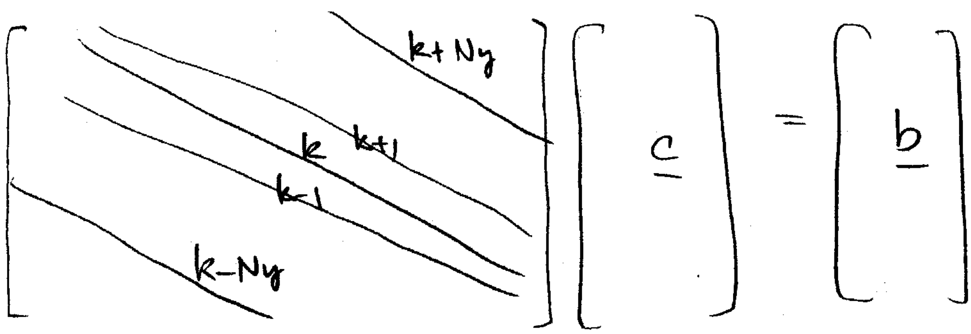


Difference Equations

$$\frac{C_{k+N_y} - 2C_k + C_{k-N_y}}{\Delta x^2} + \frac{C_{k+1} - 2C_k + C_{k-1}}{\Delta y^2} = 0$$

for $k = 1, \cdots N_x * N_y$
(exclude B.C's)

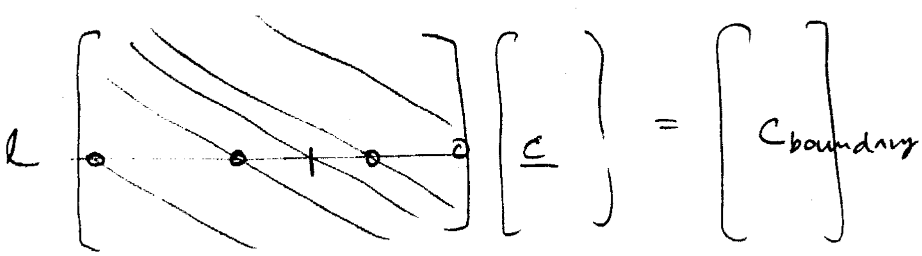
Matrix.



$$\underline{\underline{A}}\ \underline{c} = \underline{b}$$

Insert BC's

$C_l = C_{boundary}$



set $A_{ll} = 1$
$A_{lj} = 0 \quad j \neq l$
$b_l = C_{boundary}$

Solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$

① Directly : Banded Matrix

② Iteratively :

— Symmetric

— Diagonally dominant

For $\Delta x = \Delta y$

Diagonal term $\left|\frac{4}{\Delta x^2}\right|$

Off diagonal tem $\left|\frac{1}{\Delta x^2}\right|$

$\longrightarrow$ Convergence of iterative method

(2) Galerkin FEM in 2-D

Procedure

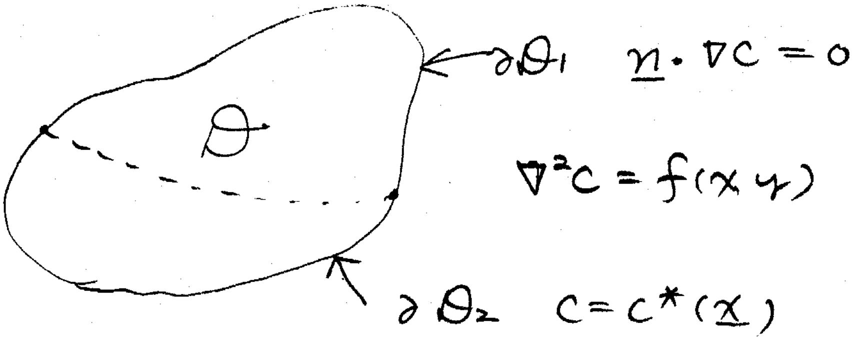
i) Discretization into sub-domain or elements

- triangular
- quadrilateral

- discretization is not unique
- Boundary is fit only approximately

ii) Approximation

iii) Form Galerkin Residual Equation



Assume

$$c(x, y) = \tilde{c}^*(x, y) + \sum_{i=1}^{N} \alpha_i \Phi^i(x, y)$$

$\tilde{c}^*(x, y)$: approx $c^*(x)$ on the boundary $\partial\theta_2$

$\sum_{i=1}^{N} \alpha_i \Phi^i(x, y)$: approx $c(x, y)$ in the interior and on $\partial\theta_1$

$$\Phi^i(x, y) = 0 \text{ on } \partial\theta_2.$$

Galerkin method

$$\int_{\Omega} \Phi^i (\nabla^2 c - f)\, dA = 0 \qquad i = 1 \cdots N$$

$$\Phi^i \nabla^2 c = \nabla \cdot (\Phi^i \nabla c) - \nabla \Phi^i \nabla c$$

$$\int_{\Omega} \{\nabla \cdot (\Phi^i \nabla c) - \nabla \Phi^i \nabla c - \Phi^i f\}\, dA = 0$$

Divergence theorem.

$$\int_{\Omega} \nabla \cdot (\Phi^i \nabla c)\, dA = \oint_{\partial\Omega} \underline{n} \cdot (\Phi^i \nabla c)\, dl$$

$$= \oint_{\partial\Omega} \Phi^i \underbrace{(n \cdot \nabla c)}_{\text{flux}}\, dl$$

$$= \oint_{\partial\Omega_1} \Phi^i \cancelto{0}{(n \cdot \nabla c)}\, dl + \oint_{\partial\Omega_2} \cancelto{0}{\Phi^i} (n \cdot \nabla c)\, dl$$

If $n \cdot \nabla c = q^*$ on $\partial\Omega_1$, replace

$n \cdot \nabla c = q^*$

On $\partial\Omega_2$, $\Phi^i = 0$.

$$\therefore \int_{\Omega} \nabla \Phi^i \nabla c\, dA = - \int_{\Omega} \Phi^i f\, dA.$$

$$\nabla = \frac{\partial}{\partial x}\underline{e}_x + \frac{\partial}{\partial y}\underline{e}_y$$

$$\int_{\theta}\left(\frac{\partial \Phi^i}{\partial x}\frac{\partial c}{\partial x} + \frac{\partial \Phi^i}{\partial y}\frac{\partial c}{\partial y}\right) dA = -\int_{\theta} \Phi^i f(x,y)\, dA$$

forcing function

$$\frac{\partial c}{\partial x} = \frac{\partial \tilde{c}^*(x,y)}{\partial x} + \sum_{j=1}^{N} \alpha_j \frac{\partial \Phi^j}{\partial x}$$

$$\frac{\partial c}{\partial y} = \frac{\partial \tilde{c}^*(x,y)}{\partial y} + \sum_{j=1}^{N} \alpha_j \frac{\partial \Phi^j}{\partial y}$$

Substitute

$$\sum_{j=1}^{N} \alpha_j \int_{\theta}\left(\frac{\partial \Phi^i}{\partial x}\frac{\partial \Phi^j}{\partial x} + \frac{\partial \Phi^i}{\partial y}\frac{\partial \Phi^j}{\partial y}\right) dA$$

$$= -\int_{\theta}\left(\frac{\partial \Phi^i}{\partial x}\frac{\partial \tilde{c}^*}{\partial x} + \frac{\partial \Phi^i}{\partial y}\frac{\partial \tilde{c}^*}{\partial y} + \Phi^i f\right) dA$$

$$\Rightarrow \quad \underline{\underline{A}}\,\underline{\alpha} = \underline{b}$$

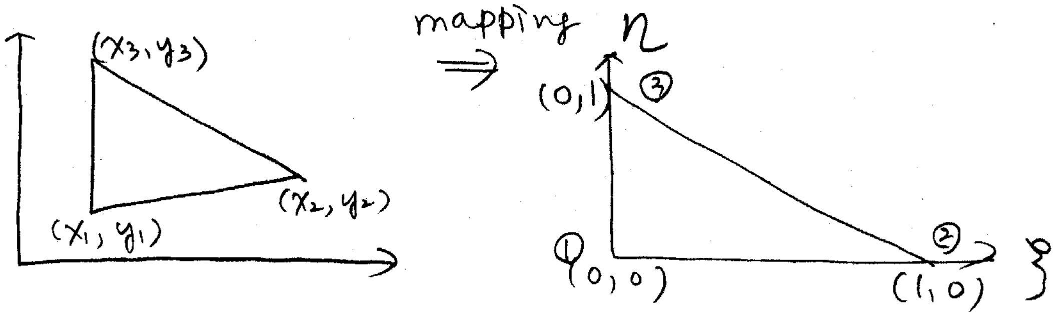
$$A_{ij} = \int_{\theta}\left(\frac{\partial \Phi^i}{\partial x}\frac{\partial \Phi^j}{\partial x} + \frac{\partial \Phi^i}{\partial y}\frac{\partial \Phi^j}{\partial y}\right) dA$$ stiffness matrix

symmetric

We cannot say about the structure of A_{ij} until we define basis function.

Elements

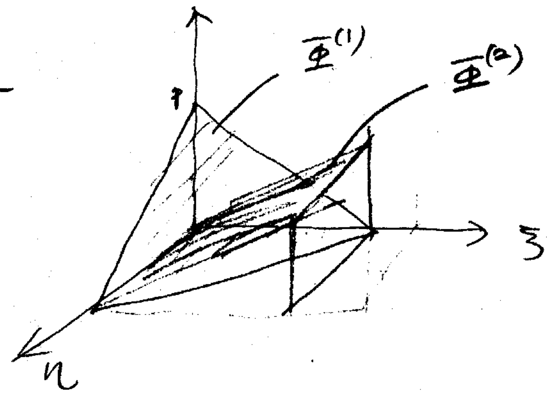
Triangular



Define

$$\Phi^{(1)}(\xi,\eta) = 1-\xi-\eta$$
$$\Phi^{(2)}(\xi,\eta) = \xi$$
$$\Phi^{(3)}(\xi,\eta) = \eta$$

$$\Phi^{i}(\xi_j,\eta_j) = \delta_{ij}$$



Iso parametric mapping

$$x = \sum_{i=1}^{3} x_i \Phi^{(i)}(\xi,\eta) \qquad y = \sum_{i=1}^{3} y_i \Phi^{(i)}(\xi,\eta)$$

$$\begin{bmatrix} \dfrac{\partial \Phi^i}{\partial x} \\ \\ \dfrac{\partial \Phi^i}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \Phi^i}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial \Phi^i}{\partial \eta}\dfrac{\partial \eta}{\partial x} \\ \\ \dfrac{\partial \Phi^i}{\partial \xi}\dfrac{\partial \xi}{\partial y} + \dfrac{\partial \Phi^i}{\partial \eta}\dfrac{\partial \eta}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \Phi^i}{\partial \xi} \\ \\ \dfrac{\partial \Phi^i}{\partial \eta} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial \Phi^i}{\partial \xi} \\[2ex] \dfrac{\partial \Phi^i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \Phi^i}{\partial x}\dfrac{\partial x}{\partial \xi} + \dfrac{\partial \Phi^i}{\partial y}\dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial \Phi^i}{\partial x}\dfrac{\partial x}{\partial \eta} + \dfrac{\partial \Phi^i}{\partial y}\dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \Phi^i}{\partial x} \\[2ex] \dfrac{\partial \Phi^i}{\partial y} \end{bmatrix}$$

Invert

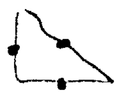
$$\rightarrow \begin{bmatrix} \dfrac{\partial \Phi^i}{\partial x} \\[2ex] \dfrac{\partial \Phi^i}{\partial y} \end{bmatrix} = \frac{1}{\left(\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi}\right)} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\[2ex] -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \Phi^i}{\partial \xi} \\[2ex] \dfrac{\partial \Phi^i}{\partial \eta} \end{bmatrix}$$

Do integral by numerical quadrature

$$\int_0^1 \int_0^{1-\xi} f(\xi,\eta)\, d\eta\, d\xi = \Delta \sum_{k=1}^{N} W_k f(\xi_k,\eta_k)$$

$(\xi_k,\eta_k) = \left(\frac{1}{3},\frac{1}{3}\right)$, $W_k = 1$ $O(h^2)$

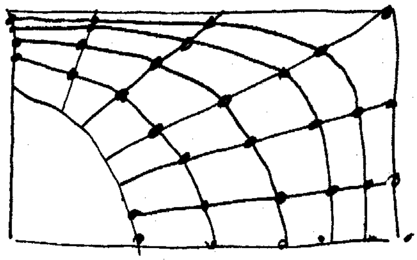
$$\left.\begin{matrix} \left(\frac{1}{2},\frac{1}{2}\right) & W_k = 1/3 \\ \left(0,\frac{1}{2}\right) & 1/3 \\ \left(\frac{1}{2},0\right) & 1/3 \end{matrix}\right\} O(h^3)$$

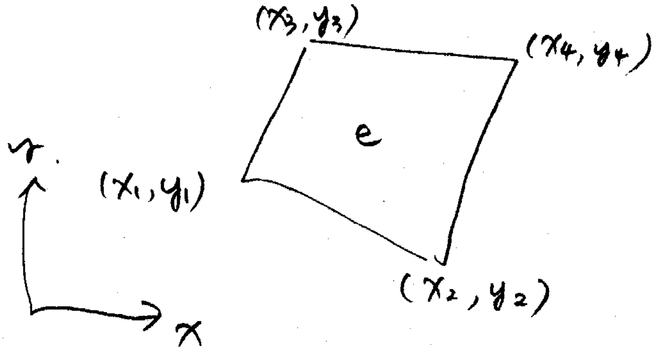


$$\begin{matrix} \left(\frac{1}{3},\frac{1}{3}\right) & \frac{27}{60} \\ \left.\begin{matrix}\left(\frac{1}{2},\frac{1}{2}\right) \\ \left(0,\frac{1}{2}\right) \\ \left(\frac{1}{2},0\right)\end{matrix}\right\} & \frac{8}{60} \\ \left.\begin{matrix}(1,0) \\ (0,1) \\ (0,0)\end{matrix}\right\} & \frac{3}{60} \end{matrix}$$

Δ : area of triangle $\quad 2\Delta = \det \begin{vmatrix} 1 & \xi_1 & \eta_1 \\ 1 & \xi_2 & \eta_2 \\ 1 & \xi_3 & \eta_3 \end{vmatrix}$

Quadrilateral Elements





Primitive Basis

$$u^e(x,y) = a_1 + a_2 x + a_3 y + a_4 xy$$

bilinear polynomial $\neq (a_1' + a_2' x)(a_3' + a_4' y)$

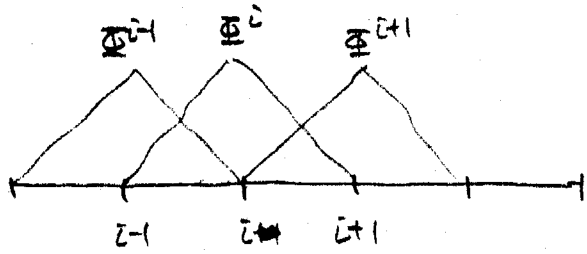
Cardinal Form of Basis

$$\Phi^i(x,y) = a_1^i + a_2^i x + a_3^i y + a_4^i xy$$

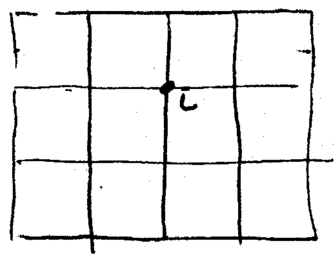
in each element
for each node (i-th)

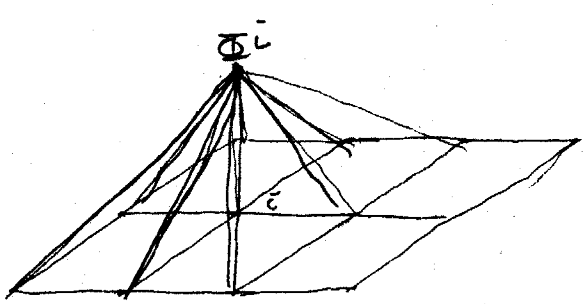
$$\Phi^i(x_j, y_j) = \delta_{ij}$$

1-D



2-D





Function Shape.

Define bilinear Lagrangian Interpolation

$$u^e(x,y) = \sum_{i=1}^{N} \underbrace{\alpha_i}_{u(x_i,y_i)} \Phi^i(x,y)$$

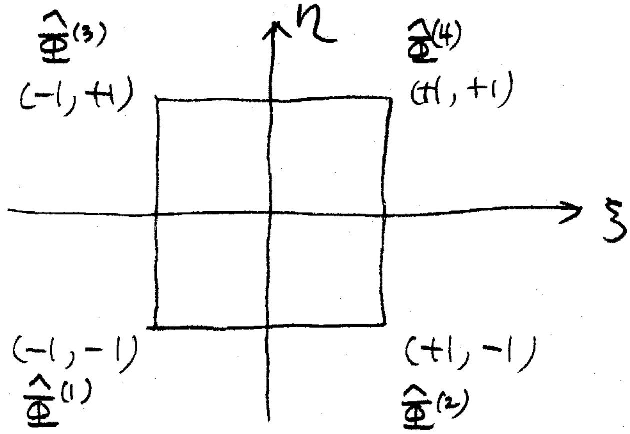
For (x_k, y_k)

$$u(x_k,y_k) = \sum_{i=1}^{N} \alpha_i \Phi^i(x_k,y_k)$$

$$= \sum_{i=1}^{N} \alpha_i \delta_{ik}$$

$$= \alpha_k$$

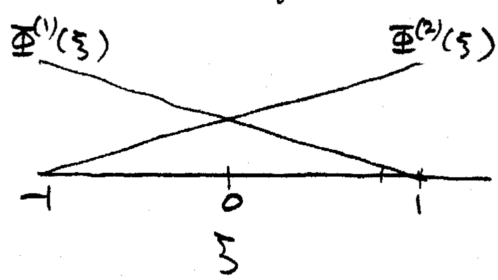
Unit Element



$$\hat{\Phi}^{(i)}(\xi,\eta) = a_1^i + a_2^i\xi + a_3^i\eta + a_4^i\xi\eta$$

$$\hat{\Phi}^{(i)}(\xi_j,\eta_j) = \delta_{ij}$$

1-D Analogue



$$\Phi^{(1)}(\xi) = \frac{1-\xi}{2}$$

$$\Phi^{(2)}(\xi) = \frac{1+\xi}{2}$$

$$\hat{\Phi}^{(1)}(\xi,\eta) \equiv \Phi^{(1)}(\xi)\,\Phi^{(1)}(\eta)$$ Tensor Product Formulation

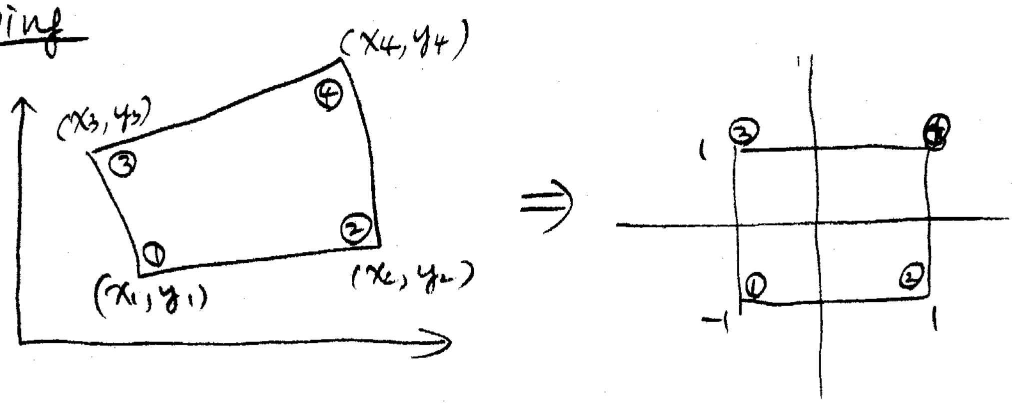
$$\hat{\Phi}^{(1)}(\xi,\eta) = \Phi^{(1)}(\xi)\,\Phi^{(1)}(\eta)$$

$$\hat{\Phi}^{(2)}(\xi,\eta) = \Phi^{(2)}(\xi)\,\Phi^{(1)}(\eta)$$

$$\hat{\Phi}^{(3)}(\xi,\eta) = \Phi^{(1)}(\xi)\,\Phi^{(2)}(\eta)$$

$$\hat{\Phi}^{(4)}(\xi,\eta) = \Phi^{(2)}(\xi)\,\Phi^{(2)}(\eta)$$

Mapping



$$x = \sum_{i=1}^{4} x_i \, \Phi^{(i)}(\xi, \eta)$$
$$y = \sum_{i=1}^{4} y_i \, \Phi^{(i)}(\xi, \eta)$$

Isoparametric Mapping

Ex. $\xi = 1, \ \eta = 1$ $\qquad x = x_4$, $y = y_4$

$\Phi^{(4)}(1,1) = 1$

$\Phi^{(i)}(1,1) = 0$ otherwise

If you want to get $\xi = \xi(x, y)$, $\eta = \eta(x, y)$

It's difficult because it's nonlinear eqn.

Sample Integral

$$P = \int_{D} \frac{\partial \Phi^i}{\partial x} \frac{\partial \Phi^j}{\partial x} \, dA = \sum_{k=1}^{N_E} \int_{D_{E_k}} \frac{\partial \Phi^i}{\partial x} \frac{\partial \Phi^j}{\partial y} \, dA$$

dxdy

$$= \sum_{k=1}^{N_E} \int_{D_u} \frac{\partial \Phi^i}{\partial x} \frac{\partial \Phi^j}{\partial y} \, |\underline{J}_{\phi}| \, dA$$

Jacobian of Mapping

$$|\underline{J}| = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi}.$$

Do integral by numerical quadrature

Numerical Integration.

$$I = \int_{\square_u} I(\xi, \eta)\, d\xi\, d\eta$$

$$= \sum_{k=1}^{N_g} I(\xi_k, \eta_k)\, W_k$$

(ξ_k, η_k) quadrature points
(gauss points)

W_k weighting function

3 point Gauss Quadrature in 1-D

$w_1 = \frac{5}{9}$ $\qquad x_1 = -\frac{\sqrt{3}}{\sqrt{5}}$

$w_2 = \frac{8}{9}$ $\qquad x_2 = 0$

$w_3 = \frac{5}{9}$ $\qquad x_3 = \sqrt{\frac{3}{5}}$

9 point Gauss Quadrature in 2-D

$\frac{5}{9} * \frac{5}{9}$	$(x, y) = -\frac{\sqrt{3}}{\sqrt{5}},$	$-\frac{\sqrt{3}}{\sqrt{5}}$
$\frac{5}{9} * \frac{8}{9}$	$-\frac{\sqrt{3}}{\sqrt{5}}$	0
$\frac{5}{9} * \frac{5}{9}$	$-\frac{\sqrt{3}}{\sqrt{5}}$	$\frac{\sqrt{3}}{\sqrt{5}}$
$\frac{8}{9} * \frac{5}{9}$	0	$-\frac{3}{\sqrt{5}}$
$\frac{8}{9} * \frac{8}{9}$	0	0
$\frac{8}{9} * \frac{5}{9}$	0	$\frac{\sqrt{3}}{\sqrt{5}}$
$\frac{5}{9} * \frac{5}{9}$	$\frac{\sqrt{3}}{\sqrt{5}}$	$-\frac{3}{\sqrt{5}}$
$\frac{5}{9} * \frac{8}{9}$	$\sqrt{\frac{3}{5}}$	0
$\frac{5}{9} * \frac{5}{9}$	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{3}{5}}$