Sensitivity analysis for FOPDT system: polynomial chaos approach

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I. INTRODUCTION

Very often, a model of a system is described by equations involve unknown parameters which must be estimated from experiment data. Among the parameters, however, only few have a small influence on the system response and thus is relative less important than the others. The study of relative important of the parameters over entire parameters spaces is termed Global Sensitivity Analysis (GSA) [1]. One approach in GSA is variance based. Generalized polynomial chaos (gPC) has been used for uncertainty quantification in a large variety of engineering problems [2-6]. It involves expressing stochastic quantities as orthogonal polynomials of the random input parameters; various orthogonal polynomials can be chosen to achieve better convergence. In this work we are interested in the GSA using Sobol's indices to determines parameter mostly responsible quantitatively of the uncertain in the output of a first order plus time delay system (FOPDT).The Sobol's indices are obtained from the ANOVA decomposition of the output. Several numerical examples are presented to illustrate the efficiency of the computational of Sobol's indices by gPC method over the existing methods such as quasi Monte- Carlo (QMC) and Monte-Carlo (MC).

II. PROBLEM FORMULATION

Consider a FOPDT system in a closed loop feedback configuration with a PID controller. Due to randomness of its parameters, the output is random process given by the solution of differential equation

$$y(t,\boldsymbol{\xi}) = f(t,\xi_1,\xi_2,...,\xi_n), \tag{1}$$

where $\boldsymbol{\xi} = (\xi_1,\xi_2,...,\xi_n)$ are unknown parameters, considered as independent random variables with given distributions $\prod_{i=1}^{n}\rho_i(\xi_i)$.

Mean and variance of y can be computed as

$$\mu_y(t) = \mathrm{E}(y(t,\xi)) = \int_{\Gamma_1}\cdots\int_{\Gamma_n} f(t,\xi_1,\xi_2,...,\xi_n)\prod_{i=1}^{n}\rho_i(\xi_i)d\xi_1\cdots d\xi_n, \tag{2}$$

$$D_y(t) = \mathrm{E}\left[(y(t)-\mu_y(t))^2\right] = \int_{\Gamma_1}\cdots\int_{\Gamma_n}\left[f(t,\xi_1,\xi_2,...,\xi_n)-\mu_y(t)\right]^2\prod_{i=1}^{n}\rho_i(\xi_i)d\xi_1\cdots d\xi_n, \tag{3}$$

where Γ_i is support domain of random variable ξ_i.

The output of system model can be decomposed into summands of increasing dimension [6]

$$y(t,\boldsymbol{\xi}) = f_0 + \sum_{i=1}^{n} f_i(t,\xi_i) + \sum_{i=1}^{n-1}\sum_{i<j}^{n} f_{i_j}(t,\xi_i,\xi_j) + \cdots + f_{1,...,n}(t,\xi_1,...,\xi_n) \tag{4}$$

The term $f_0 = \mu_y(t)$ is the mean value.

The 2^n term in Eq. (4) are given by

$$f_i(t,\xi_i) = \mathrm{E}[y(t,\boldsymbol{\xi})\,|\,\xi_i] - f_0(t)$$

$$f_{ij}(t,\xi_i,\xi_j) = \mathrm{E}[y(t,\boldsymbol{\xi})\,|\,\xi_i,\xi_j] - f_i(t,\xi_i) - f_j(t,\xi_j) - f_0(t), \tag{5}$$

…

where $\mathrm{E}[y(t,\boldsymbol{\xi})\,|\,\xi_i]$ is the conditional expectation of $y(t,\boldsymbol{\xi})$ when ξ_i is set.

The terms are orthogonal to each other. The decomposition of (4) lead to the following decomposition of the variance [7]

$$D_y(t) = \sum_{i=1}^{n} D_i(t) + \sum_{i=1}^{n-1}\sum_{i<j}^{n} D_{ij}(t) + \cdots + D_{1,\cdots,n}(t)\,, \tag{6}$$

with

$$D_i(t) = Var\left[\mathrm{E}[y(t,\boldsymbol{\xi}) \mid \xi_i]\right], \tag{7}$$

$$D_{ij}(t) = Var\left[\mathrm{E}[y(t,\boldsymbol{\xi}) \mid \xi_i, \xi]\right] - D_i(t) - D_j(t)$$

$$\vdots$$

$$D_{1,\cdots,n}(t) = D_y(t) - \sum_{i=1}^{n} D_i(t) - \sum_{1\le i<j\le n} D_{ij}(t) - \cdots - \sum_{1\le i_1<\cdots<i_{n-1}\le n} D_{i_1\cdots i_{n-1}}(t)$$

where $Var\left[\mathrm{E}[y(t,\boldsymbol{\xi}) \mid \xi_i]\right]$ is the conditional expectation when ξ_i is set.
Thus, the first order sensitivity function is defined as follows

$$S_i = \frac{D_i(t)}{D(t)} \tag{8}.$$

The first order sensitivity function represents the main effect of the parameter ξ_i which corresponds to its contribution to the output alone. The closer to 1 the value of $S_i(t)$ is, the more parameter ξ_i contributes to the total variances of the output [1].
The gPC approach provides computational effective method for estimated statistics (e.g mean and variance) of complex quantities. Its algorithm is briefly below.

- Construct a suitable *n* dimensional cubature set $\left\{\boldsymbol{\xi}^{(\mathbf{j})}, w^{(\mathbf{j})}\right\}_{\mathbf{j}=1}^{Q}$, with a suitable ordered index **j.**
- For each $\mathbf{j} = 1,\ldots,Q$, obtain system output $y(t,\boldsymbol{\xi}^{(\mathbf{j})})$.
- The solution's mean is estimated as

$$\mu_y(t) = \sum_{\mathbf{j}=1}^{Q} y(t,\boldsymbol{\xi}^{(\mathbf{j})})\mathbf{w}^{(\mathbf{j})} = \sum_{j_1=1}^{q_1}\cdots\sum_{j_d=1}^{q_n} y(\xi_1^{(j_1)},\ldots,\xi_n^{(j_n)})(w_1^{(j_1)}\ldots w_n^{(j_n)}) \tag{9}$$

- The variance of the solution is estimated as

$$D_Y(t) \approx \sum_{\mathbf{j}=1}^{Q} (y(t,\boldsymbol{\xi}^{(\mathbf{j})}) - \mu_y(t))^2\,\mathbf{w}^{(\mathbf{j})} \tag{10}.$$

where $\mu_y(t)$ is approximated as in (9)

The conditional means and variances in Eqs. (7) and (8) are estimated in a similar manner.

Eq (7) is computationally impractical in MC and QMC frame. For this reason the below algorithm detailed the practical computation of the Sobol's indices by MC and QMC method.

- Let us consider a sample set of *N* realizations of the random components. The mean and variance of output are estimated as

$$\hat{\mu}_y(t) = f_0(t) = \frac{1}{N}\sum_{i=1}^{N} f(t,\boldsymbol{\xi}^{(i)})\,, \tag{11}$$

$$\hat{D}_y(t) = \frac{1}{N}\sum_{i=1}^{N} f^2(t,\boldsymbol{\xi}^{(i)}) - \hat{D}_0(t) \tag{12}$$

where $\hat{D}_0(t) = f_0(t)^2$.

- The conditional variance is given by

$$D_j = \int_{\Gamma_j} \left\{ \int_{\Gamma_i} \cdots \int_{\Gamma_n} f(\xi_1, \cdots, \tilde{\xi}_j, \cdots, \xi_n) \prod_{\substack{i=1 \\ i \neq j}}^{n} \rho_i(\xi_i) d\xi_i \right\}^2 \rho_j(\tilde{\xi}_j) d\tilde{\xi}_j - \hat{\mu}_y^{\,2} \qquad (13).$$

The integral in (13) can be recasted as [7]

$$\begin{aligned} &\int \cdots \int f(\xi_1, \cdots, \xi_j, \cdots, \xi_n) f(\nu_1, \cdots, \xi_j, \cdots, \nu_n) \prod_{i=1}^{n} \rho_i(\xi_i) d\xi_i \prod_{\substack{i=1 \\ i \neq j}}^{n} \rho_i(\nu_i) d\nu_i \\ &\approx \frac{1}{N} \sum_{i=1}^{N} f(\xi_1^{(i)}, \cdots, \xi_j^{(i)}, \cdots, \xi^{(i)}{}_n) f(\nu_1^{(i)}, \cdots, \xi_j^{(i)}, \cdots, \nu^{(i)}{}_n) \end{aligned} \qquad (14).$$

while the estimated computational cost of (13) is $\mathrm{O}(N^2)$, the computational cost for Eq. (14) is only $\mathrm{O}(2N)$. Thus, Eq. (14) is more practical for estimated the Sobol's indices than Eq. (13)

III. EXAMPLES

a. Example 1

Consider a FOPDT system

$$G(s) = \frac{K}{Ts+1} e^{-s}, \qquad (15)$$

where the K and T are uniform variables in [0.5,1.5]. The system is in closed loop feedback configuration with a PID controller $C(s) = 0.5932 + 0.5213 / s + 0.1294s$. The input is a unit step. The Sobol's indices estimated by gPC, MC, and QMC are shown in Figure 1. The total computational times and simulation parameters for each method is given in the table 1. From table 1 it can be seen that the gPC can give similar results with far less computational effort than the MC and QMC method. Note that for the MC and QMC method the conditional variances are computed from Eq. (14). The Halton's sequence is used for QMC method. It also can be seen that the convergence of the sensitivity index estimate is significant improve by the use of QMC sequence. From this Fig. it can be seen that the effect of uncertain K is quite simple. On the contrary, the sensitivity function of the time constant is seemed to oscillatory.

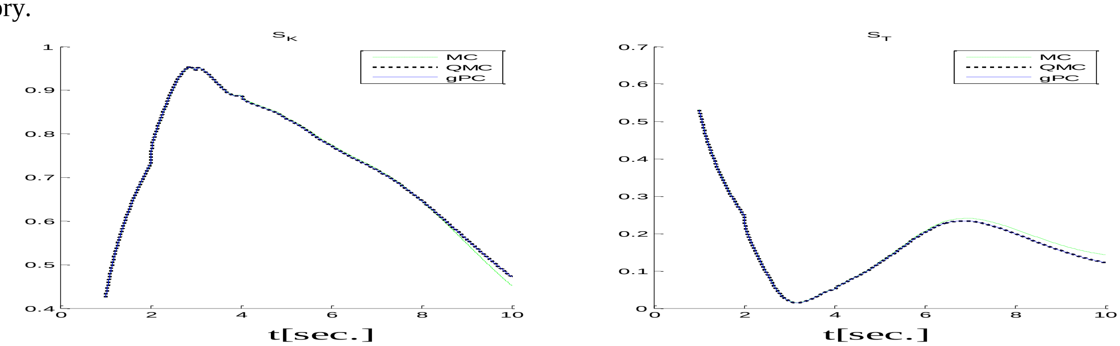

Figure 1. Dynamic sensitivity indices for example 1.

b. Example 2

Consider the FOPDT system with where K and T are now of triangular distribution Tr(0.5,1,1.5). The triangular distribution is given by the following distribution functions

$$Tr(a,c,b) = \begin{cases} \dfrac{(x-a)^2}{(b-a)(c-a)} & \text{if } a \le x \le c \\[2ex] 1 - \dfrac{(b-x)^2}{(b-a)(b-c)} & \text{if } c \le x \le b \end{cases} \qquad (16)$$

The same PID controller as the previous example was used. The Sobol 's indices estimated by gPC, MC, and QMC are shown in Figure 2. Since the previous example has already demonstrated the computational effectiveness of the proposed gPC and QMC methods over the MC method, the results by the gPC are compared with the QMC (Halton sequence) method in this example. The computational times for different methods for this example also were given in Table 1.the The MATLAB suite OPQ [8] can be used to obtain one-dimensional quadrature sets and their corresponding orthogonal polynomials (polynomial chaos) with respect to triangular weights in this example. By comparison of Fig.1 and Fig.2, it can be seen that the relative importance of the time constant has increased while the distribution has changed from uniform to triangle distribution.

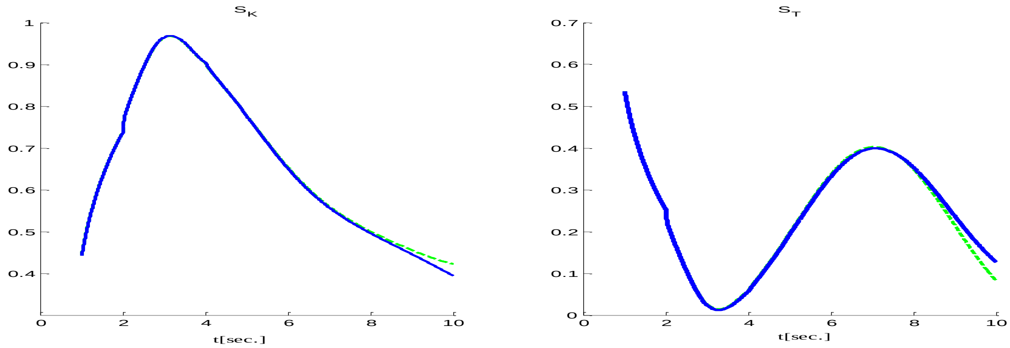

Figure 2. Dynamic sensitivity indices for example 2.

Table 1. Computational time and Simulation parameters

Example	Simulation parameters			Computational time (sec.)		
	MC	QMC	Proposed	MC	QMC	Proposed
1	50000 Samples	8000 Samples	125 Cubature nodes	27375.89	1968.47	13.89
2	N/A	8000 Samples	125 Cubature nodes	N/A	1852.31	14.26

IV. CONCLUSION

This paper presented an alternative approach for computing the dynamic sensitivity indices by polynomial chaos method as the dynamic of the system changes over time. Two numerical examples with FOPDT systems are presented to illustrate the efficiency of the gPC method over the existing methods (e.g. QMC, MC).The method also can be made available for arbitrary kind distribution by construct its corresponding polynomial chaos .However, the application of the method is entirely condition on the availability of the gPC expansions. For smooth functions, where the gPC method allows for spectral convergence, the method is expected to outperform the traditional methods (MC, QMC). Since the method is based on the gPC expansion, it also suffers from the limitations of gPC method. The first is curse of dimensionality due to the exponential growth of the basis dimension with the number of independent random components [4].The second is the loss of spectral convergence for non-smooth functions. The approach presents good results for relative simple dynamic FOPDT models with two parameter uncertainties only. Further works will be carried out on model with higher number of uncertainties than the model presented in this work.

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REFERENCES

[1] A. Satelli et.al. . *Global Sensitivity Analysis: The primer*,Wiley, 2008.
[2] K.A. Puvkov, N.D. Egupov, A.M. Makarenkov, A.I. Trofimov. *Theory and Numerical Methods for Studying Stochastic Systems*. Fizmatlits: Moscow, 2003.
[3] P.L.T.Duong M. Lee. Robust PID controller design for processes with stochastic parametric uncertainties. *J. Process Control*, In Press.
[4] D. Xiu. *Numerical method for stochastic computations: A spectral method approach*, Princeton University Press, 2010.
[5] O.P. Lemaitre, O.M. Knio. *Spectral Methods for Uncertainty Quantification: With Application to Computational Fluid Dynamics*, Springer Science+Business Media, Dordrecht,2010.
[6] E. H. Sandoval, F. A. Collin, M. Basset. Sensitivity study of dynamic systems using polynomial chaos. *Reliability Engineering and System Safety*, 2012, 104:15-26,2012
[7] I.M. Sobol. Sensitivity estimates for nonlinear mathematical models. *Mathematical Modelling and Computing Experiments* 1993,1 :407–14.
[8] W. Gautschi. *Orthogonal Polynomials: Computation and Approximation*, Oxford University Press, 2004..