Uncertainty quantification for fractional order system: Polynomial chaos approach

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I. Introduction

Stochastic uncertainty may arise in systems when the physics governing the system is known and the system parameters are either not known precisely or expected to vary in the operational lifetime. Such uncertainty also occurs when the system models are built from experimental data using system identification techniques, where a system plant is represented by its transfer function with unknown parameters. As a result, the values of the parameters in the transfer function have a range of uncertainty. In order to include this uncertainty in the mathematical model, various probabilistic methods have been developed. Traditional probabilistic approaches to uncertainty quantification include the Monte Carlo method [1,2] and its variants such as Latin Hypercube Sampling [3] which generate ensembles of random realizations for the prescribed random inputs and use repetitive deterministic solvers for each realization. Although such methods are straightforward to apply, their convergence rates can be relatively slow. For example, the mean value typically converges as $1/\sqrt{K}$, where K is the number of realization. The need for a large number of samples for accurate results causes an excessive computational burden.

The recently developed stochastic generalized polynomial chaos (gPC) methods exhibit faster convergence for problems with relatively large random uncertainties. In the gPC methods, stochastic solutions are expressed as orthogonal polynomial of the input random uncertainties. In this work, the gPC are using for uncertainty quantification.

II. Fractional order system

Fractional integral Rieman Liouville integral of function $f(t)$ is defined by [4]:

$$(I_a^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{1}$$

Among several formulas of the generalized derivative, the most common used one is the Riemann-Liouville definition :

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} (\frac{d}{dt})^m \int_a^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}} d\tau \tag{2}$$

where m is the integer satisfied $m-1<\alpha<m$

For the generalized integration and differentiation, the property of linearity, similarly to the integer case, is conserved.

Fractional linear models

Laplace transform of fractional order differentiation is defined as:

$$? \frac{d^{\alpha} f(t)}{dt^{\alpha}}\} = s^{\alpha} F(s) - \sum_{k=0}^{m-1} f^k(0^+) s^{\alpha-1-k} \tag{3}$$

In particular, if the derivatives of the function $f(t)$ are all equal to 0 at, Eq. (3) can be rewritten as:

$$? \frac{d^{\alpha} f(t)}{dt^{\alpha}}\} = s^{\alpha} F(s) \tag{4}$$

The fractional order linear system of single variablescan be defined as

$$G(s) = \frac{Y(s)}{R(s)} = \frac{b_m s^{\beta_m} + ... + b_0 s^{\beta_0}}{s^{\alpha_n} + ... + a_0 s^{\alpha_0}} \tag{5}$$

The orders α_i, β_i are arbitrary real positive, $r(t)$ and $y(t)$ are respectively the input and output of system.

III. STOCHASTIC ANALYSIS OF FRACTIONAL ORDER SYSTEM [5]

Let us consider the fractional systems governed by equations 5 with:

$$\alpha_i = \alpha_i(\xi), \beta_j = \beta_j(\xi) \tag{6}$$

Where $\xi = (\xi_1, \xi_2, ..., \xi_N)$ are a random vector of mutually independent random component with probability density function $\rho_i(\xi)$, y denotes state variable.

A. Polynomial chaos theory

In the gPC method, one seeks an approximate of response function $f(y(t,\xi))$ via an orthonormal polynomial of random variables:

$$f^P(y(t,\xi)) = \sum_{i=1}^{M} \hat{f}_m(t)\Phi_m(\xi);$$
$$M+1 = \begin{pmatrix} N+P \\ N \end{pmatrix} \tag{7}$$

where P is the order of polynomial chaos, $\hat{f}_m$ the coefficient of the gPC expansion satisfying eq. (8) as:

$$\hat{f}_m = E[\Phi_m f(y)] = \int_\Gamma f(y)\Phi_m(\xi)\rho(\xi)d\xi \tag{8}$$

where E[] denotes the expectation.

B. Stochastic collocation

Stochastic collocation approach can easily deal with complex response functions. Its algorithm is described below:

- Choose a collocation set $\left\{\xi^{(m)}, w^{(m)}\right\}_{m=1}^{Q}$ for the random vector ξ , where $\xi^{(m)} = (\xi_1^{(m)}, ..., \xi_N^{(m)})$ is the mth node and *w(m)* is the corresponding weight.
- For each node, solve eq. (5) to obtain its solution $\tilde{y}^{(m)} = \tilde{y}(t, \xi^{(m)})$ and evaluate the response function $\tilde{f}^{(m)}$.
- Calculate the approximation of the gPC coefficients via a discrete integration rule for eq. (8).

$$\tilde{\tilde{f}}_j = \ell^Q[\tilde{f}(y,\xi)\Phi(\xi)] = \sum_{m=1}^{Q} \tilde{f}^{(m)}\Phi_j(\xi^{(m)})w^{(m)} \quad j = 1, ..., M \tag{9}$$

- Construct the N-variate, Pth order gPC approximation of response function

$$\tilde{f}_N^P = \sum_{j=1}^{M} \tilde{\tilde{f}}_j(\xi)\Phi_j(\xi) \tag{10}$$

The collocation set $\left\{\xi^{(m)}, w^{(m)}\right\}_{m=1}^{Q}$ should be chosen in such a way that an accurate integration can be constructed, i.e., for a smooth function $g(\xi)$:

$$\ell^Q[g] = \sum_{m=1}^{Q} g(\xi^{(m)})w^{(m)} \approx \int_\Gamma g(\xi)\rho(\xi)d\xi \tag{11}$$

In the classical spectral method [4], Gaussian quadrature [5] is chosen as one dimensional (1D) numerical integration rule. The multi-dimensional numerical integration can be constructed by tensorization of 1D quadrature rule $Q_{q_i}^{(1)}$:

$$\ell^Q[g] = (Q_{q_1}^{(1)} \otimes ... \otimes Q_{q_N}^{(1)})g \tag{12}$$

where the subscript in $Q_{q_i}^{(1)}$ denotes the number of node for 1D quadrature rule.

C. Statistical analysis of fractional order system

Once all the gPC coefficients are evaluated by a numerical method, a post-processing procedure is carried out to obtain the statistics.

The mean value is the first expansion coefficient:

$$E[\tilde{f}_N^P] = \mu_f = \int_\Gamma \tilde{f}_N^P \rho(\xi)d\xi = \int_\Gamma \left[\sum_{j=1}^{M} \tilde{\tilde{f}}_j(\xi)\Phi_j(\xi)\right]\rho(\xi)d\xi = \tilde{\tilde{f}}_1 \tag{13}$$

The variance of response function f(y) is evaluated as:

$$D_f = \sigma^2{}_f = E[(f-\mu_f)^2] = \int_\Gamma (\sum_{j=1}^{M} \tilde{\tilde{f}}_j(\xi)\Phi_j(\xi) - \tilde{\tilde{f}}_1)(\sum_{j=1}^{M} \tilde{\tilde{f}}_j(\xi)\Phi_j(\xi) - \tilde{\tilde{f}}_1)\rho(\xi)d\xi = \sum_{j=2}^{M} \tilde{\tilde{f}}_j^2 \tag{14}$$

In eqs. (13) and (14), the property that the polynomial set starts with $\Phi_1(\xi)=1$ is employed. Furthermore, the weight function of the polynomial is the probability density function. If the response function is chosen by $f(y)=y$, the mean and variance of system output are approximately given by eqs. (13) and (14), respectively.

The set $\{\phi_i\}_{i=1}^{d_i}$ is the orthonormal polynomial of ξ_i with the weight function $\rho_i(\xi_i)$, which is the probability density function of a random variable ξ_i. This establishes a correspondence between the distribution of the random variable ξ_i and the type of the orthonornal polynomial of it gPC basis. Here, we consider only uniform and Gaussian stochastic uncertainties and the corresponding generalized Legendre polynomial chaos. For details of other types stochastic uncertainties and their corresponding gPC basis, see [5, 6] and the references therein.

IV. CASE STUDIES

Consider the fractional order system [7]:

$$G(s)=\frac{1}{0.8s^{\alpha}+0.5s^{\beta}+1} \tag{15}$$

in closed loop feedback with the controller $C(s)=20.5+3.73s^{1.15}$. The order α,β are random variables with types of distribution are given in table 1. Monte Carlo and gPC are applied for predicting the mean and variance of system output. Figure 1shows mean and variances of system output. The fractional order systems were integrated by the matlab code fode_sol [4]. Calculations were made using the library DEMM [6]. Since the variance is a measure of variability of random process, a large variance of system output under uncertainties in the order implies a large deviation from nominal response.

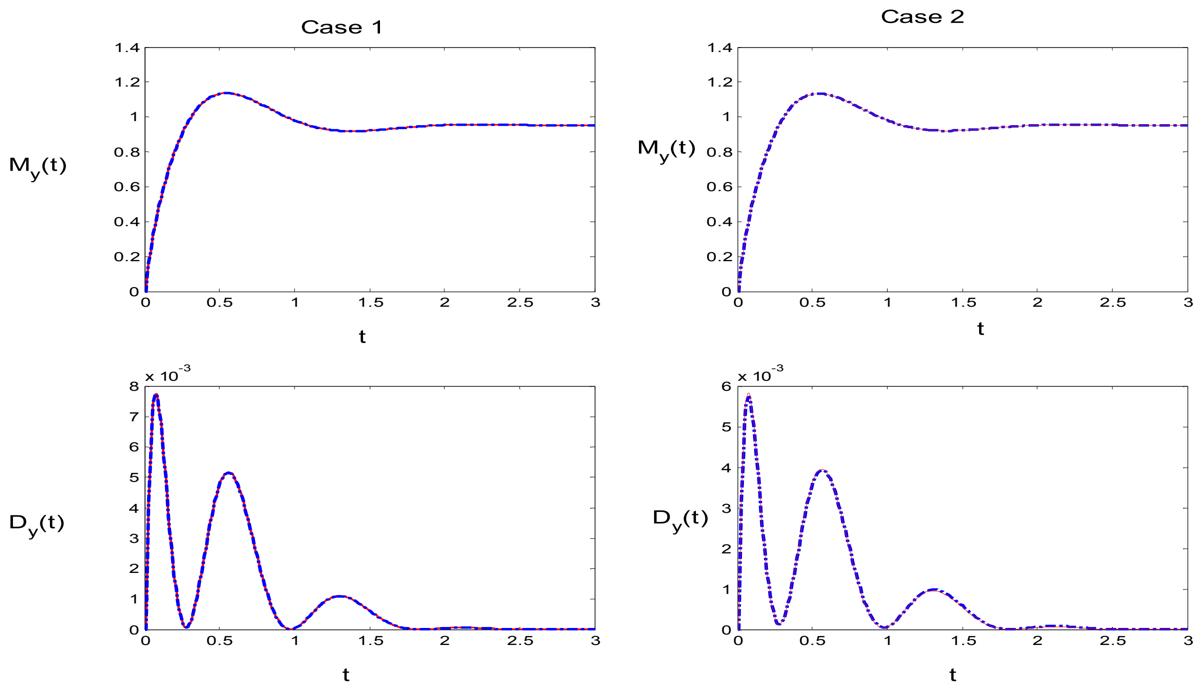

Figure 1 Mean and variances of system output. Monte Carlo: Solid red line; gPC: Blue dash dot line.

Case	**Monte Carlo**		**gPC**
1) $\alpha \in [1.8, 2.2], \in \beta[0.8, 0.9]$	Number of Samples: 2500	Comp. time 38.14 s.	Comp .time 0.2 s.
2) $\alpha \in \mathbf{N}(2.0.01), \beta \in \mathbf{N}(0.85, 0.01)$	Number of Samples: 2500	Comp. time 32.14 s.	Comp .time 0.2 s.

Table 1 Simulation parameters and computational time.

V. CONCLUSIONS

In this work, a stochastic analysis for fractional order system with random order was studied. Simulation studies have shown that the method gives accurate results for prediction statistical characteristic of fractional order system with random order. It is shown that the use of gPC method drastically reduces a computation time with a desired accuracy over that by the traditional Monte-Carlo method. Hence, the gPC may be more suitable than Monte Carlo for uncertainty quantification for fractional order system with random order under the restriction of short-time integration and random space regularness.

REFERENCES

[1] K. A. Puvkov, N. D. Egupov, A. M. Makarenkov, *Theory and Numerical Methods for Studying Stochastic Systems*, Moscow, Fizmatlits, 2003[in Russian].
[2] J. S. Liu, *Monte Carlo Strategies in Scientific Computing*, Springer-Verlag, 2001.
[3] M. Stein, Large Sample properties of simulation using Latin-Hypercube sampling, *Technometrics*, 29(2) , 1987 143–151.
[4] C. A. Monje, Y. Q. Chen, B. M .Vinagre, D .Xue, V .Feliu, *Fractional-order Systems and Controls: Fundamentals and applications*, Springer 2010.
[5] D. Xiu, *Numerical method for stochastic computations: A spectral method approach*, Princeton university press, 2010.
[6] K. A. Puvkov, N. D. Egupov, A. M. Makarenkov, *Theory and Numerical Methods for Studying Stochastic Systems.* Moscow, Fizmatlits, 2003[in Russian].
[7] N.Tan, O.F .Ozguven, M. M. Ozyetkin, Robust stability analysis of fractional order interval polynomial, *ISA Transaction,* 48*,* 2009, 166-172.