Linear Systems

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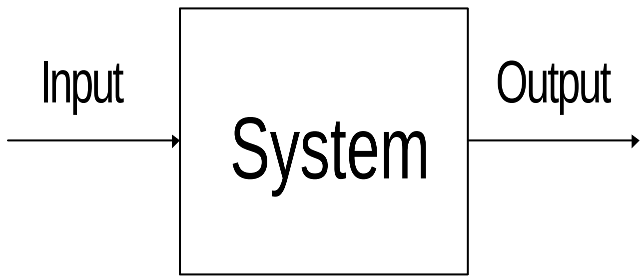
Chapter 1

Introduction

Signal: a quantitative phenomenon that varies with time.
System: a signal processor



$$y(t) = \mathcal{L}\, u(t)$$

Classification of Systems

1. Causal
   - causal (physical, nonanticipative)
   - noncausal (anticipative)
2. Deterministic

- deterministic
- probabilistic (stochastic)

3. Linear
   - linear: satisfy the principle of superposition

$$\mathcal{L}(c_1u_1 + c_2u_2) = c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2)$$

   - nonlinear

4. Stationary
   - stationary (static, instantaneous, memoryless)
   - dynamic

5. Lumped Parameter
   - lumped parameter (finite order) systems: described by a finite number of variables [Ex: system described by ODE's]
   - distributed parameter (infinite order) systems: described by function variables [Ex: system described by PDE's]

6. Continuous
   - continuous (Ex: system described by differential equs.)
   - discrete (Ex: system described by difference equs.)

7. SISO and MIMO

Chapter 2

Mathematical Preliminaries

2.1 Linear Space

Linear (or Vector) Space V over a scalar Field $\mathcal{F}$: set with the following axioms:

axioms for linear space

- Addition

$$+ : V \times V \to V : (x, y) \to x + y;$$

  1. Associative

  $$(x + y) + z = x + (y + z)$$

  2. Commutative

  $$x + y = y + x$$

  3. $\exists$ identity 0 such that

  $$x + 0 = 0 + x = x$$

  4. $\exists$ inverse $-x$ such that

  $$x + (-x) = 0$$

- Scalar Multiplication:

$$\cdot : \mathcal{F} \times V \to V : (a, x) \to ax;$$

1. Associative

$$(ab)x = a(bx)$$

- Distributive law:

$$(a+b)x = ax + bx$$
$$a(x+y) = ax + ay$$

Linear Space Examples
Canonical Example I:
$\mathcal{F}^n$: n-tuples $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathcal{F}$
Addition and Scalar Multiplication

$$x + y = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \quad ax = \begin{bmatrix} ax_1 \\ \vdots \\ ax_n \end{bmatrix}$$

Ex: $\mathbf{R}^n$
Canonical Example II:
Function space F: all functions $f(d)$ from a domain D to V
Addition and Scalar Multiplication

$$(f+g)(d) = f(d) + g(d)$$
$$(af)(d) = af(d)$$

Ex: L_p Set of Lebesgue measurable (integrable) functions such that

$$||f||_p = \left(\int |f(t)|^p \right)^{\frac{1}{p}} < \infty \quad 1 \leq p < \infty$$
$$||f||_\infty = ess \sup_t |f(t)| < \infty$$

Subspace: A subset of linear space that forms a linear space by itself.
Ex: the set of all vectors in $\mathbf{R}^n$ whose first component is zero.

Definition: the family of vectors $\{v_i\}_{i=1}^n \subset V$ are linearly independent iff

$$a_1 v_1 + \cdots + a_n v_n = [v_1 \ \cdots \ v_n] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = 0$$

implies

$$a_1 = \cdots = a_n = 0 \quad \Leftrightarrow \quad \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = 0.$$

$$\Downarrow$$

The family of vectors $\{v_i\}_{i=1}^n \subset V$ are linearly dependent iff $\exists$ $\{a_i\}_{i=1}^n$, not all zero such that

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

Def.: The maximal number of linearly independent vectors in a linear space is called the dimension of the linear space.

Ex: $dim(\mathbf{R}^n) = n$, $dim(\mathbf{R}^{n\times n}) = n^2$.

Definition: the family of vectors $\{b_i\}_{i=1}^n \subset V$ is said to be a basis if elements of the family are linearly independent each other and

$$V = Sp(\{b_i\}_{i=1}^n) := \left\{ \sum_{i=1}^n a_i b_i : a_i \in \mathcal{F} \right\}.$$

The elements of the family are called basis vectors of V.

Note: Suppose $\{b_i\}_{i=1}^n$ is a basis of V. If $x \in V$, $\exists$ unique $\{\xi_i\}_{i=1}^n$ such that

$$x = \sum_{i=1}^n \xi_i b_i.$$

If not unique, b_i's are not linearly independent $(0 = x - x = \sum_{i=1}^n (\xi_i - \xi_i')b_i)$.

$\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$ is called the component vector (or representation) of x w.r.t. the basis $\{b_i\}_{i=1}^n$.

Theorem: In a n-dimensional vector space, any set of n linearly independent vectors qualifies as a basis.

Proof: Let $\{u_i\}_{i=1}^n$ be the set of n linearly independent vectors. Then $\{x\} \cup \{u_i\}_{i=1}^n$ is linearly dependent. Hence

$$a_0 x + a_1 u_1 + \cdots + a_n u_n = 0$$

where not all a_i's are zero. This implies $a_0 \neq 0$ since $\{u_i\}_{i=1}^n$ is linearly dependent otherwise. Then

$$x = b_1 u_1 + \cdots + b_n u_n$$

where $b_i = -\frac{a_i}{a_0}$.

2.2 Linear Operators

U, V: linear spaces over $\mathcal{F}$

A: operator from U to V

Def.: A is linear if

$$A(a_1 u_1 + a_2 u_2) = a_1 A u_1 + a_2 A u_2$$

Fact: $0 \in \mathcal{N}(A)$.

Proof: $A0 = A(0 \cdot u) = 0Au = 0$.

Ex: $U = L_1 = \{ f(t) : \int_0^\infty |f(t)| dt < \infty \}$, $V = \mathbf{R}$

$$Au = \int_0^\infty u(t) dt$$

Null Space (Kernel):

$$\mathcal{N}(A) = \{u \in U : Au = 0\}$$

Range Space (Image):

$$\mathcal{R}(A) = \{v \in V : v = Au, u \in U\} = AU$$

Fact: $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are linear subspaces.

Proof: Let $x_1, x_2 \in \mathcal{N}(A)$. Then $Ax_1 = Ax_2 = 0$. By linearity,

$$A(ax_1 + bx_2) = aAx_1 + bAx_2 = 0.$$

$\Rightarrow\ ax_1 + bx_2 \in \mathcal{N}(A)$.

Let $y_1, y_2 \in \mathcal{R}(A)$. Then $\exists x_1, x_2 \in U$ such that $y_1 = Ax_1$ and $y_2 = Ax_2$. Then $ax_1 + bx_2 \in U$. By linearity,

$$A(ax_1 + bx_2) = aAx_1 + bAx_2 = ay_1 + by_2 \in \mathcal{R}(A).$$

Theorem: A is injective (one-to-one) iff $\mathcal{N}(A) = \{0\}$.

Proof: ($\Rightarrow$) Obvious.

($\Leftarrow$) Suppose the contrary. Then $\exists\ x \neq y$ such that $Ay = Ax \Rightarrow A(y - x) = 0 \Rightarrow y = x$ (contradiction).

Facts:

1. If $\{Au_i\}$ is a linearly independent family, then so is $\{u_i\}$.

2. The converse of the above holds iff A is injective.

Proof: 1)

$$a_1u_1 + \cdots + a_nu_n = 0.$$
$$\Downarrow$$
$$a_1Au_1 + \cdots + a_nAu_n = A(a_1u_1 + \cdots + a_nu_n) = 0.$$
$$\Downarrow$$
$$a_1 = \cdots = a_n = 0.$$

2) ($\Leftarrow$)

$$0 = a_1Au_1 + \cdots + a_nAu_n = A(a_1u_1 + \cdots + a_nu_n).$$

By one-to-one assumption,

$$a_1u_1 + \cdots + a_nu_n = 0.$$
$$\Downarrow$$
$$a_1 = \cdots = a_n = 0.$$

($\Rightarrow$) Suppose the contrary. Then $\exists\ x \neq y$ such that $Ay = Ax$. Notice that x, y linearly independent ($x = ay$ is not possible) $\Rightarrow$ Ax, Ay linearly independent (contradiction).

Theorem: Suppose $dim U = dim V = n$. Then TFAE

1. A is injective ($\mathcal{N}(A) = \{0\}$).

2. A is surjective (onto) ($\mathcal{R}(A) = V$).

3. A is bijective (injective+surjective).

4. A^{-1} exists.

Proof: ($1 \Rightarrow 2$) From the previous fact 2, $\{u_i\}$ is a basis implies $\{Au_i\}$ is so.

$$V = \{a_1 Au_i + \cdots + a_n Au_n : a_i \in \mathcal{F}\} = \{A(a_1 u_i + \cdots + a_n u_n) : a_i \in \mathcal{F}\}$$
$$= \{Au : u \in U\} = \mathcal{R}(A).$$

($1 \Leftarrow 2$) Let $\{v_i\}$ be a basis of V.

A surjective $\Rightarrow \exists\, u_i$ such that $v_i = Au_i \Rightarrow \{u_i\}$ is a basis of U.

Consider $\{a_i\}$ not all a_i's are zero. Then $a_1 u_1 + \cdots + a_n u_n \neq 0$ implies $A(a_1 u_1 + \cdots + a_n u_n) = a_1 Au_1 + \cdots + a_n Au_n = a_1 v_1 + \cdots + a_n v_n \neq 0$. Hence, $\mathcal{N}(A) = \{0\}$.

($2 \Leftrightarrow 3$) Obvious from the equivalence of 1) and 2).

($3 \Rightarrow 4$) inverse mapping is well defined.

($3 \Leftarrow 4$) Notice that, if A^{-1} exists, $Ax_1 = Ax_2$ implies $x_1 = x_2$. Hence, injective.

2.3 Matrix Representation

Let $\{u_j\}_{j=1}^n$ be the basis for U. Then

$$x = \sum_{j=1}^n \xi_j u_j.$$

By linearity of A,

$$Ax = A\sum_{j=1}^n \xi_j u_j = \sum_{j=1}^n \xi_j Au_j.$$

Let $\{v_i\}_{i=1}^m$ be the basis for V. Then

$$Au_j = \sum_{i=1}^m a_{ij} v_i.$$

$$\Downarrow$$

$$\sum_{i=1}^{m}\eta_i v_i = y = Ax = \sum_{j=1}^{n}\xi_j A u_j = \sum_{j=1}^{n}\xi_j \sum_{i=1}^{m} a_{ij} v_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\xi_j\right) v_i.$$

By uniqueness of representation,

$$\eta = A\xi$$

where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

Theorem: Let $\{u_j\}_{j=1}^{n}$ and $\{v_i\}_{i=1}^{m}$ be the bases for U and V, respectively. Then, w.r.t. these bases, A is represented by the $m \times n$ matrix.

Change of Basis

Let $\{u_k\}_{k=1}^{n}$ and $\{\tilde{u}_i\}_{i=1}^{n}$ be two bases for U and $\{v_k\}_{k=1}^{m}$ and $\{\tilde{v}_i\}_{i=1}^{m}$ two bases for V. Then

$$\tilde{u}_i = \sum_{k=1}^{n} p_{ki} u_k.$$

$$\Downarrow$$

$$\sum_{k=1}^{n}\xi_k u_k = x = \sum_{i=1}^{n}\tilde{\xi}_i \tilde{u}_i = \sum_{i=1}^{n}\tilde{\xi}_i\left(\sum_{k=1}^{n} p_{ki} u_k\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} p_{ki}\tilde{\xi}_i\right) u_k.$$

$$\Downarrow$$

$$\xi = P\tilde{\xi}.$$

Notice that the ith column of P is the representation of $\tilde{u}_i$ w.r.t $\{u_j\}$.

Similarly,

$$\tilde{\eta} = Q\eta.$$

Notice that the ith column of Q is the representation of v_i w.r.t $\{\tilde{v}_j\}$.

Let $y = Ax \Rightarrow \eta = A\xi \Rightarrow$

$$\tilde{\eta} = QA\xi = QAP\tilde{\xi}.$$

$$\Downarrow$$

the representation of linear operator w.r.t. $\{\tilde{u}_i\}$ and $\{\tilde{v}_i\}$ is

$$\tilde{A} = QAP.$$

Special Case: $V = U$ and use same basis for both domain and range. Then

$$\xi = P\tilde{\xi} = PQ\xi \;\Rightarrow\; PQ = I \;\Rightarrow\; Q = P^{-1} \;\Rightarrow\; \tilde{A} = P^{-1}AP.$$

Such transformation from A to $\tilde{A}$ is called similarity transformation.

Range and Null Spaces

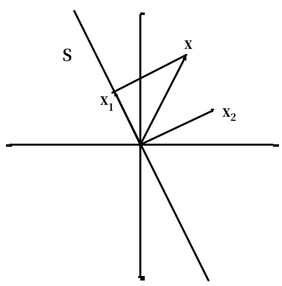
Fact: Let $A = [a_1 \cdots a_n]$. Then $\mathcal{R}(A) = Sp(\{a_j\})$.

Let S be a subspace of $\mathbf{R}^n$. Then the orthogonal complement of S is defined as

$$S^\perp := \{x \in \mathbf{R}^n : \langle x, y\rangle = 0,\ \forall y \in S\}.$$

Fact (Orthogonal Decomposition): $\mathbf{R}^n = S \oplus S^\perp$.

Proof: Suppose $x \in \mathbf{R}^n$. Let x_1 be the projection of x on S. Then $x_2 = x - x_1$ is orthogonal to S and thus $x_2 \in S^\perp$.



Lemma: $\mathcal{R}(A)$ is the orthogonal complement of $\mathcal{N}(A^*)$

Proof:

$$y \in [\mathcal{R}(A)]^\perp \quad \Leftrightarrow \quad y^* z = 0\ \forall z \in \mathcal{R}(A)$$

$$\Leftrightarrow \quad (A^* y)^* x = y^* A x = 0\ \forall x \in \mathbf{R}^n \quad \Leftrightarrow \quad A^* y = 0 \quad \Leftrightarrow \quad y \in \mathcal{N}(A^*).$$

From this lemma,

$$\mathbf{R}^m = \mathcal{R}(A) + \mathcal{N}(A^*).$$

Similarly,

$$\mathbf{R}^n = \mathcal{R}(A^*) + \mathcal{N}(A).$$

Fact:

$$\mathcal{R}(A) = \mathcal{R}(AA^*) \quad \text{and} \quad \mathcal{R}(A^*) = \mathcal{R}(A^*A).$$

Proof:

$$\mathcal{R}(A) = \{Ax : x \in \mathbf{R}^n\} = \{Ax : x \in \mathcal{R}(A^*) + \mathcal{N}(A)\}$$

$$= \{A(x+z) : x \in \mathcal{R}(A^*),\ z \in \mathcal{N}(A)\} = \{Ax : x \in \mathcal{R}(A^*)\}$$

$$= \{AA^*y : y \in \mathbf{R}^n\} = \mathcal{R}(AA^*).$$

Similarly, the second equality also follows.

Definition: The rank (nullity) of the $m \times n$ matrix $A = dim\mathcal{R}(A)$ $(dim\mathcal{N}(A))$.

Fact: $rank(A) + nullity(A) = n = dim\mathcal{D}(A)$

Proof: Let $\{u_i\}_{i=1}^k$ be the basis of $\mathcal{N}(A)$. Complete that basis such that $\{u_i\}_{i=1}^n$ is the basis of $\mathbf{R}^n$. Then $x = \sum_{i=1}^n \xi_i u_i$ and

$$Ax = A\left(\sum_{i=1}^n \xi_i u_i\right) = \sum_{i=1}^k \xi_i Au_i + \sum_{i=k+1}^n \xi_i Au_i.$$

$Au_i = 0$, $i = 1, \cdots, k$ because $u_i \in \mathcal{N}(A)$.

$\Rightarrow$ $\{Au_i\}_{i=k+1}^n$ spans $\mathcal{R}(A)$.

Claim: $\{Au_i\}_{i=k+1}^n$ is a linearly independent family. Assume the contrary. Then $\exists\ a_{k+1}, \cdots, a_n$ (not all zero) such that

$$0 = \sum_{i=k+1}^n a_i Au_i = A\left(\sum_{i=k+1}^n a_i u_i\right).$$

$\Rightarrow$ $\sum_{i=k+1}^n a_i u_i \in \mathcal{N}(A) \Rightarrow$ contradiction and the claim follows.

$\{Au_i\}_{i=k+1}^n$ is a basis for $\mathcal{R}(A)$ with $dim\mathcal{R}(A) = n - k$.

One may conjecture that $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are disjoint. But this is not the case.

Ex: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $\mathcal{N}(A) = \mathcal{R}(A) = Span(e_1)$.

Fact:

1. $0 \leq rank(A) \leq \min\{m, n\}$

2. $rank(A)$ is equal to

   (a) maximum number of linearly independent columns of A

   (b) maximum number of linearly independent rows of A

Proof: Notice that $\mathcal{R}(A)$ is the span of the columns of A.
1) If $n \geq m$, $rank(A) \leq m$.
If $n \leq m$, $rank(A) \leq n$.
2) a) Obvious from above.
b) Since $\mathbf{R}^n = \mathcal{R}(A^*) + \mathcal{N}(A)$, $rank(A^*) = n - nullity(A) = rank(A)$.

Corollary: Suppose A is an $n \times n$ matrix. Then

$$rank(AA^*) = n \quad \Leftrightarrow \quad rank(A) = n \quad \Leftrightarrow \quad rank(A^*) = n \quad \Leftrightarrow \quad rank(A^*A) = n.$$

Sylvester Inequality: A $m \times n$ matrix, B $n \times p$ matrix

$$rank(A) + rank(B) - n \leq rank(AB) \leq \min\{rank(A), rank(B)\}.$$

Proof: Since $\mathcal{R}(AB) \subset \mathcal{R}(A)$,

$$rank(AB) \leq rank(A).$$

Moreover the domain of A in AB is $\mathcal{R}(B)$ and from the above fact 1)

$$rank(AB) \leq rank(B).$$

Thus, the second inequality follows.

Similar to $rank(A) + nullity(A) = n$, we can show $rank(AB) = rank(B) - d$ where d is the dimension of $\mathcal{R}(B) \cap \mathcal{N}(A)$. However, $dim\mathcal{N}(A) = n - rank(A)$ and thus the first equality follows.

Fact: A $m \times n$ matrix

$$rank(AC) = rank(A) \quad \text{and} \quad rank(DA) = rank(A)$$

for any $n \times n$ and $m \times m$ nonsingular matrices C and D.

Proof: Proof follows from Sylvester inequality.

2.4 Spectral Theory and Jordan Representation of Square Matrices

Def: $\lambda \in \mathbf{C}$ is called an eigenvalue of A if $\exists$ right (left) eigenvector $x(y) \neq 0$ such that $Ax = \lambda x$ ($y^* A = \lambda y^*$).

Fact: λ is an eigenvalue of A iff it is a solution of the characteristic polynomial

$$\chi_A(\lambda) = det(\lambda I - A) = 0.$$

The eigenvector x is a nonzero vector in $\mathcal{N}(\lambda I - A)$.

Theorem: Let $\lambda_1, \cdots, \lambda_n$ be the distinct eigenvalues of A and v_i be an eigenvector associated with λ_i. Then $\{v_i\}_{i=1}^n$ is linearly independent.

Proof: Suppose the contrary. $\exists\, a_i$'s (not all zero) such that

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

WLOG, we assume $a_1 \neq 0$. Consider

$$(A - \lambda_2 I) \cdots (A - \lambda_n I) \left(\sum_{i=1}^{n} a_i v_i \right) = 0.$$

Notice that

$$(A - \lambda_j I) v_i = (\lambda_i - \lambda_j) v_i \quad \text{if } j \neq i$$

and

$$(A - \lambda_i I) v_i = 0.$$

Hence,

$$a_1 (\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_n) v_1 = 0.$$

Since λ_i's are distinct, this implies $a_1 = 0$ (contradiction!).

Def.: A matrix is simple if it has n linearly independent eigenvectors.

Corollary: If eigenvalues of A are all distinct, A is simple.

Remark: There exist simple matrices whose eigenvalues of A are not all distinct. (Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$)

Let A be simple. Define

$$V = [v_1 \ \cdots \ v_n] \qquad \Lambda = diag\{\lambda_1, \cdots, \lambda_n\}.$$

Then

$$AV = V\Lambda.$$

Since V is nonsingular, we have

$$V^{-1}AV = \Lambda.$$

Note that Λ is the representation of A w.r.t. its eigenvectors.

Fact: If A is simple, A can be diagonalized by similarity transform.

Def: A vector v is said to be a generalized eigenvector of grade k of A associated with λ if

$$(A - \lambda I)^k v = 0$$

and

$$(A - \lambda I)^{k-1} v \neq 0.$$

Let v be a generalized eigenvector of grade k of A.

$$v_k = v$$

$$v_{k-1} = (A - \lambda I)v = (A - \lambda I)v_k$$
$$v_{k-2} = (A - \lambda I)^2 v = (A - \lambda I)v_{k-1}$$
$$\vdots$$
$$v_1 = (A - \lambda I)^{k-1} v = (A - \lambda I)v_2.$$

Notice that v_i is a generalized eigenvector of grade i.

Let $\mathcal{N}_i = \mathcal{N}[(A - \lambda I)^i]$. Then $\mathcal{N}_i \subset \mathcal{N}_{i+1}$. Indeed,

$$(A - \lambda I)^i v_i = (A - \lambda I)^i (A - \lambda I)^{k-i} v = (A - \lambda I)^k v = 0$$

and

$$(A - \lambda I)^{i-1} v_i = (A - \lambda I)^{i-1} (A - \lambda I)^{k-i} v = (A - \lambda I)^{k-1} v \neq 0$$

$$\Downarrow$$

$$v_i \in \mathcal{N}_i \quad \text{but} \quad v_i \notin \mathcal{N}_{i-1}.$$

Let A have eigenvalue λ with multiplicity m.

Question: find m linearly independent generalized eigenvectors associated with λ.

1. compute the ranks of $(\boldsymbol{A}-\lambda\boldsymbol{I})^i$ until $rank(\boldsymbol{A}-\lambda\boldsymbol{I})^k = n-m$.

   For simplicity, assume $n = 10$, $m = 8$, $k = 4$ and

i	$rank(\boldsymbol{A}-\lambda\boldsymbol{I})^i$	$dim\mathcal{N}_i$
0	10	0
1	7	3
2	4	6
3	3	7
4	2	8

2. Because $\mathcal{N}_3 \subset \mathcal{N}_4$ and $dim\mathcal{N}_4 - dim\mathcal{N}_3 = 1$, $\exists$ one and only one linearly independent vector u such that

$$u \in \mathcal{N}_4 \quad \text{but} \quad u \notin \mathcal{N}_3$$

$$\Downarrow$$

$$(\boldsymbol{A}-\lambda\boldsymbol{I})^4 u = 0 \quad \text{but} \quad (\boldsymbol{A}-\lambda\boldsymbol{I})^3 u \neq 0.$$

   Let

$$u_1 = (\boldsymbol{A}-\lambda\boldsymbol{I})^3 u \quad u_2 = (\boldsymbol{A}-\lambda\boldsymbol{I})^2 u \quad u_3 = (\boldsymbol{A}-\lambda\boldsymbol{I}) u \quad u_4 = u.$$

3. Because $\mathcal{N}_2 \subset \mathcal{N}_3$ and $dim\mathcal{N}_3 - dim\mathcal{N}_2 = 1$, u_3 is the one and only one linearly independent vector such that

$$u_3 \in \mathcal{N}_3 \quad \text{but} \quad u_3 \notin \mathcal{N}_2.$$

4. Because $\mathcal{N}_1 \subset \mathcal{N}_2$ and $dim\mathcal{N}_2 - dim\mathcal{N}_1 = 3$, in addition to u_2, $\exists$ two more linearly independent vectors v and w such that

$$v, w \in \mathcal{N}_2 \quad \text{but} \quad v, w \notin \mathcal{N}_1$$

$$\Downarrow$$

$$(\boldsymbol{A}-\lambda\boldsymbol{I})^2 v = 0 \quad \text{but} \quad (\boldsymbol{A}-\lambda\boldsymbol{I}) v \neq 0$$

$$(\boldsymbol{A}-\lambda\boldsymbol{I})^2 w = 0 \quad \text{but} \quad (\boldsymbol{A}-\lambda\boldsymbol{I}) w \neq 0.$$

   Let

$$v_1 = (\boldsymbol{A}-\lambda\boldsymbol{I}) v \quad v_2 = v$$

$$w_1 = (\boldsymbol{A}-\lambda\boldsymbol{I}) w \quad w_2 = w.$$

5. $\mathcal{N}_0 \subset \mathcal{N}_1$ and $dim\mathcal{N}_1 - dim\mathcal{N}_0 = 3$. Notice that u_1, v_1, w_1 are three vectors such that

$$u_1, v_1, w_1 \in \mathcal{N}_1 \quad \text{but} \quad u_1, v_1, w_1 \notin \mathcal{N}_0.$$

Theorem: The generalized eigenvectors generated above are linearly independent.

Proof: Suppose u_1, v_1, w_1 are linearly dependent. Then, $\exists\ c_1, c_2, c_3$, not all zero,

$$c_1 u_1 + c_2 v_1 + c_3 w_1 = 0.$$

However,

$$0 = c_1 u_1 + c_2 v_1 + c_3 w_1 = (\boldsymbol{A} - \lambda \boldsymbol{I})(c_1 u_2 + c_2 v + c_3 w) =: (\boldsymbol{A} - \lambda \boldsymbol{I}) y.$$

Hence $y \in \mathcal{N}_1$. However, $u_2, v, w \in \mathcal{N}_2 \setminus \mathcal{N}_1$ such that $\mathcal{N}_1 + span\{u_2, v, w\}$ $\Rightarrow$ $y \in \mathcal{N}_1 \cap span\{u_2, v, w\}$ $\Rightarrow$ $y = 0$ $\Rightarrow$ u_2, v, w are linearly dependent. By contraposition, u_1, v_1, w_1 are linearly independent because u_2, v, w are linearly independent.

Suppose $u_1, u_2, u_3, u_4, v_1, v_2, w_1, w_2$ are linearly dependent. Then $\exists\ \{c_i\}_{i=1}^{8}$, not all zero,

$$c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4 + c_5 v_1 + c_6 v_2 + c_7 w_1 + c_8 w_2 = 0.$$

Multiplying $(\boldsymbol{A} - \lambda \boldsymbol{I})^3$, $c_4 (\boldsymbol{A} - \lambda \boldsymbol{I})^3 u_4 = 0 \Rightarrow c_4 = 0$.

Multiplying $(\boldsymbol{A} - \lambda \boldsymbol{I})^2$, $c_3 (\boldsymbol{A} - \lambda \boldsymbol{I})^2 u_3 = 0 \Rightarrow c_3 = 0$.

Multiplying $(\boldsymbol{A} - \lambda \boldsymbol{I})$, $c_2 (\boldsymbol{A} - \lambda \boldsymbol{I}) u_2 + c_6 (\boldsymbol{A} - \lambda \boldsymbol{I}) v_2 + c_8 (\boldsymbol{A} - \lambda \boldsymbol{I}) w_2 = c_2 u_1 + c_6 v_1 + c_8 w_2 = 0 \Rightarrow c_2 = c_6 = c_8 = 0$.

Now $c_1 u_1 + c_5 v_1 + c_7 w_1 = 0 \Rightarrow c_1 = c_5 = c_7 = 0$. (contradiction!)

Theorem: The generalized eigenvectors of $\boldsymbol{A}$ associated with different eigenvalues are linearly independent.

This theorem can be proven similar to the previous theorem by applying $(\boldsymbol{A} - \lambda_i \boldsymbol{I})^k (\boldsymbol{A} - \lambda_j \boldsymbol{I})^l$ repetitively.

Jordan form $\hat{\boldsymbol{A}}$ of $\boldsymbol{A}$: $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ = representation of $\boldsymbol{A}$ w.r.t. $\boldsymbol{P} = [u_1\ u_2\ u_3\ u_4\ v_1\ v_2\ w_1\ w_2\ *\ *]$ where $*$'s denote generalized eigenvectors associated with other eigenvalues.

First four columns of $\hat{\boldsymbol{A}}$: $\boldsymbol{P}^{-1}\boldsymbol{A} u_i$

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) u_1 = 0 \quad (\boldsymbol{A} - \lambda \boldsymbol{I}) u_2 = u_1 \quad (\boldsymbol{A} - \lambda \boldsymbol{I}) u_3 = u_2 \quad (\boldsymbol{A} - \lambda \boldsymbol{I}) u_4 = u_3$$

$$\Downarrow$$

$$Au_1 = \lambda u_1 = P\begin{bmatrix}\lambda\\0\\0\\0\\\vdots\\0\end{bmatrix} \quad Au_2 = u_1 + \lambda u_2 = P\begin{bmatrix}1\\\lambda\\0\\\vdots\\0\end{bmatrix}$$

$$Au_3 = u_2 + \lambda u_3 = P\begin{bmatrix}0\\1\\\lambda\\0\\0\\\vdots\\0\end{bmatrix} \quad Au_4 = u_3 + \lambda u_4 = P\begin{bmatrix}0\\0\\1\\\lambda\\0\\\vdots\\0\end{bmatrix}.$$

Proceeding similarly,

$$\hat{A} = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & & \\ & & & & & & & & * & * \\ & & & & & & & & & * \end{bmatrix}.$$

Cayley Hamilton Theorem: Let $\chi_A(\lambda) = det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ be the characteristic polynomial of A. Then

$$\chi_A(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n = 0.$$

Proof for simple A: Let v_i be an eigenvector of A associated with eigenvalue λ_i. Then

$$\chi_A(A)V = 0$$

where $V = [v_1 \cdots v_n]$. Since V is nonsingular, $\chi_A(A) = 0$.

Remark: By Cayley-Hamilton theorem, A^k, $k \geq n$, can be expressed in terms of $\{I, A, \cdots, A^{n-1}\}$. Hence any matrix polynomial $p(A)$ can be written as

$$p(A) = p_0(t)I + p_1(t)A + p_2(t)A^2 + \cdots = \xi_0(t)I + \xi_1(t)A + \cdots + \xi_{n-1}A^{n-1}.$$

2.5 Positive Definite Hermitian Square Matrix

Def.: A is Hermitian iff $A = A^*$.

Fact: Let A be Hermitian.

1. x^*Ax is real.
2. eigenvalues of A are all real.
3. n eigenvectors exist and are all orthogonal.

Proof: 1) $(x^*Ax)^* = x^*A^*x = x^*Ax$

2) Let λ be an eigenvalue and v be the corresponding eigenvector. Then $v^*Av = \lambda v^*v$. Note that LHS is real and v^*v is real and > 0.

3) (Proof of orthogonality) For multiple eigenvalues, we can always choose mutually orthogonal eigenvectors. Suppose $Au = \lambda u$ and $Av = \mu v$ with $\lambda \neq \mu$. Note that $u^*A = \lambda u^*$. Hence

$$u^*Av = \lambda u^*v \quad \text{and} \quad u^*Av = \mu u^*v$$

$\Rightarrow \lambda u^*v = \mu u^*v \Rightarrow u^*v = 0$.

Def.: A is positive semidefinite (PSD) if $x^*Ax \geq 0$ for all x.

Def.: A is positive definite (PD) if $x^*Ax > 0$ for all $x \neq 0$.

Fact: TFAE

1. A is PSD (PD).
2. all its eigenvalues are nonnegative (positive).

Proof: ($1 \Rightarrow 2$) Let λ_i be an eigenvalue and v_i be the corresponding unit eigenvector. Then

$$Av_i = \lambda_i v_i \quad \Rightarrow \quad 0 \leq (<) v_i^*Av_i = \lambda_i v_i^* v_i = \lambda_i.$$

$(2 \Rightarrow 1)$ $\{v_i\}$ orthonormal eigenvectors

$$Ax = A(a_1 v_1 + \cdots + a_n v_n) = a_1 A v_1 + \cdots + a_n A v_n = a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n$$

$$\Downarrow$$

$$x^* A x = (a_1 v_1^* + \cdots + a_n v_n^*)(a_1 \lambda_1 v_1 + \cdots + a_n \lambda_n v_n) = a_1^2 \lambda_1 + \cdots + a_n^n \lambda_n \geq (>) 0.$$

Fact: If A is Hermitian, $\mathcal{N}(A)$ is the orthogonal complement of $\mathcal{R}(A)$.
Proof: Notice that $\mathcal{N}(A)$ is the orthogonal complement of $\mathcal{R}(A^*) = \mathcal{R}(A)$.

2.6 Normed Linear Spaces

Norm: size of a vector in a linear space.

Norm $||\cdot||$ is a mapping from V to $\mathbf{R}^+$ satisfying
1. $||x|| = 0$ iff $x = 0$
2. $||ax|| = |a|\,||x|| \quad \forall a \in \mathbf{R}, \forall x \in V$
3. $||x + y|| \leq ||x|| + ||y|| \quad \forall x, y \in V$

Norms for $\mathbf{R}^n$

p norms:

$$||x||_p = \left(|x_1|^p + \cdots + |x_n|^p\right)^{\frac{1}{p}} \qquad 1 \leq p < \infty$$
$$||x||_\infty = \max_i |x_i|.$$

$||\cdot||_2$ is called the Euclidean norm.

Norms for $\mathbf{R}^{m\times n}$

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \in \mathbf{R}^{m\times n}$$

p norms:

$$|||A|||_p = \left(\sum_{i,j} |A_{i,j}|^p\right)^{\frac{1}{p}} \qquad 1 \leq p < \infty$$

$$|||A|||_\infty = \max_{i,j} |A_{i,j}|.$$

What is the difference between $\mathbf{R}^{m\times n}$ and $\mathbf{R}^{mn}$?

A matrix in $\mathbf{R}^{m\times n}$ defines a linear operator from $\mathbf{R}^n$ to $\mathbf{R}^m$; $y = Ax$.

induced (or operator) p norms:

$$||A||_p = \sup_{x\neq 0} \frac{||Ax||_p}{||x||_p} = \max_{||x||=1} ||Ax||_p \quad 1 \leq p \leq \infty$$

$$\Downarrow$$

$$||y||_p = ||Ax||_p \leq ||A||_p ||x||_p \quad \forall x \in \mathbf{R}^n.$$

Ex:

$p = 1$:

$$||A||_1 = \max_j \sum_{i=1}^m |a_{i,j}|.$$

$p = 2$:

$$||A||_2 = \sigma_{max}(A) := [\lambda_{max}(A^T A)]^{\frac{1}{2}}$$

where σ_{max} is the maximum singular value of A.

$p = \infty$:

$$||A||_\infty = \max_i \sum_{j=1}^m |a_{i,j}|.$$

Remark 1: For finite dimensional spaces such as $\mathbf{R}^n$ and $\mathbf{R}^{n\times n}$, two different norms are equivalent; there exists $a, b > 0$ such that

$$a||x||_\alpha \leq ||x||_\beta \leq b||x||_\alpha.$$

Convergent sequence: $\{x_k\}, x_k \in V$

$\lim_{k\to\infty} x_k = x$, $x \in V$ if $\lim_{k\to\infty} ||x - x_k|| = 0$.

Closed Set S: $\{x_k\}, x_k \in S$

$$\lim_{k\to\infty} x_k = x \Rightarrow x \in S.$$

Cauchy Sequence: $\{x_k\}, x_k \in V$, $\lim_{\substack{k\to\infty \\ m\to\infty}} ||x_m - x_k|| = 0$.

$$\text{Convergent} \underset{\not\Leftarrow}{\Rightarrow} \text{Cauchy.}$$

Complete Space V: Every Cauchy sequence converges to a vector in V.

Banach Space V: A complete normed linear space

Ex1: $\mathbf{R}^n$

Ex2: C_0: Set of continuous functions with the norm $||x||_C = \max_t ||x(t)||$.

Ex3: L_p: Set of Lebesque measurable (integrable) functions with the norm

$$||f||_p = \left(\int |f(t)|^p\right)^{\frac{1}{p}} \qquad 1 \le p < \infty$$

$$||f||_\infty = ess\sup_t |f(t)|$$

that is finite.

Chapter 3

Description of Linear Systems

3.1 State Space Description

Consider a physical system that is mathematically modeled by a system of coupled implicit finite order nonlinear ordinary differential equations.

$$F_1(t, y_1, \dot{y}_1, \cdots, y_1^{(p_1)}, y_2, \dot{y}_2, \cdots, y_2^{(p_2)}, \cdots, y_q^{(p_q)}, u_1, u_2, \cdots, u_m) = 0,$$

$$F_2(t, y_1, \dot{y}_1, \cdots, y_1^{(p_1)}, y_2, \dot{y}_2, \cdots, y_2^{(p_2)}, \cdots, y_q^{(p_q)}, u_1, u_2, \cdots, u_m) = 0,$$

$$\vdots$$

$$F_q(t, y_1, \dot{y}_1, \cdots, y_1^{(p_1)}, y_2, \dot{y}_2, \cdots, y_2^{(p_2)}, \cdots, y_q^{(p_q)}, u_1, u_2, \cdots, u_m) = 0.$$

Define

$$x_1 := y_1, \;\; x_2 := \dot{y}_1, \;\; \cdots, \;\; x_{p_1} := y_1^{(p_1-1)},$$

$$x_{p_1+1} := y_2, \;\; x_{p_1+2} := \dot{y}_2, \;\; \cdots, \;\; x_{p_1+p_2} := y_2^{(p_2-1)},$$

$$\vdots$$

$$x_{p_1+\cdots+p_{q-1}+1} := y_q, \; x_{p_1+\cdots+p_{q-1}+2} := \dot{y}_q, \; \cdots, \; x_{p_1+\cdots+p_{q-1}+p_q} := y_q^{(p_q-1)}.$$

Then

$$\dot{x}_1 = x_2, \;\; \dot{x}_2 = x_3, \;\; \cdots \;\; , \dot{x}_{p_1-1} = x_{p_1},$$

$$\dot{x}_{p_1+1} = x_{p_1+2}, \;\; \dot{x}_{p_1+2} = x_{p_1+3}, \;\; \cdots \;\; , \dot{x}_{p_1+p_2-1} = x_{p_1+p_2},$$

$$\vdots$$

$$\dot{x}_{\sum_{i=1}^{q-1} p_i+1} = x_{\sum_{i=1}^{q-1} p_i+2}, \dot{x}_{\sum_{i=1}^{q-1} p_i+2} = x_{\sum_{i=1}^{q-1} p_i+3}, \cdots, \dot{x}_{\sum_{i=1}^{q} p_i-1} = x_{\sum_{i=1}^{q} p_i},$$

$$F_1(t, x_1, x_2, \cdots, x_{p_1}, \dot{x}_{p_1}, \cdots, \dot{x}_{\sum_{i=1}^{q} p_i}, u_1, u_2, \cdots, u_m) = 0,$$
$$F_2(t, x_1, x_2, \cdots, x_{p_1}, \dot{x}_{p_1}, \cdots, \dot{x}_{\sum_{i=1}^{q} p_i}, u_1, u_2, \cdots, u_m) = 0,$$
$$\vdots$$
$$F_q(t, x_1, x_2, \cdots, x_{p_1}, \dot{x}_{p_1}, \cdots \dot{x}_{\sum_{i=1}^{q} p_i}, u_1, u_2, \cdots, u_m) = 0.$$

Assumption

$[F_1 \ \cdots F_q]^T = 0$ is solvable for $[\dot{x}_{p_1} \ \dot{x}_{p_1+p_2} \ \cdots \dot{x}_{p_1+p_2+\cdots+p_q}]^T$.

Then the nonlinear systems can be represented as:

$$\dot{x}_1 = f_1(t, x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m)$$
$$\dot{x}_2 = f_2(t, x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m)$$
$$\vdots$$
$$\dot{x}_n = f_n(t, x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_m)$$

or, in vector notation,

$$\dot{x} = f(t, x, u)$$

where

$$x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u := \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t, x, u) := \begin{bmatrix} f_1(t, x, u) \\ f_2(t, x, u) \\ \vdots \\ f_n(t, x, u) \end{bmatrix}.$$

State

Given a time instant t, the state of the system is the minimal information that are necessary to calculate the future response.

For ODE's, the concept of the state is the same as that of the initial condition.

$$\Downarrow$$

$$\text{State} = x(t)$$

Input

The forcing function u in ODE is called "input" in control theory.

Output

Some desired quantities that are l-dimensional vector function of (t, x, u):

$$y = g(t, x, u) \in \mathbf{R}^l.$$

Input-State Relation:

$$\dot{x} = f(t, x, u)$$

State-Output Relation:

$$y = g(t, x, u)$$

In this note, we focus on the linear time invariant systems (that is possibly obtained through linearization) that are described by

$$\dot{x} = \boldsymbol{A}x + \boldsymbol{B}u \qquad \text{State DE}$$

$$y = \boldsymbol{C}x + \boldsymbol{D}u \qquad \text{Readout Map}$$

Notice that the system is completely characterized by the matrix $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$

Fact: There exists a unique solution of the linear time-invariant state DE.

Let $\phi(t, t_0, x_0, u)$ be the solution of the state DE with IC $x(t_0) = x_0$. Then

$$x(t) = \phi(t, t_0, x_0, u) \qquad \text{State Transition Map}$$

$$y(t) = \boldsymbol{C}\phi(t, t_0, x_0, u) + \boldsymbol{D}u(t) \qquad \text{Response}$$

Properties of Solutions to State DE:

1. Linearity in (x_0, u)

$$\phi(t, t_0, a_1x_{01} + a_2x_{02}, a_1u_1 + a_2u_2) = a_1\phi(t, t_0, x_{01}, u_1) + a_2\phi(t, t_0, x_{02}, u_2)$$

2. Additive property

$$\phi(t, t_0, x_0, u) = \underbrace{\phi(t, t_0, x_0, 0)}_{\text{z. i. sol.}} + \underbrace{\phi(t, t_0, 0, u)}_{\text{z. s. sol.}}$$

3. Time invariance

$$\phi(t, t_0, x_0, u(\cdot)) = \phi(t - t_0, 0, x_0, u(\cdot + t_0))$$

Proof: 1) RHS satisfies the state DE and the IC when $x_0 = a_1 x_{01} + a_2 x_{02}$, $u = a_1 u_1 + a_2 u_2$. Then 1) follows from the uniqueness of solution.

2) is a special case of 1)

3) Obvious from the time invariance of DE and uniqueness of solution.

Remark: From 3), IC can be assumed as $x(0) = x_0$ WLOG.

Unforced Solution: State Transition Matrix

Unforced system: $u = 0$

By linearity, matrix representation theorem and time invariance, unforced (free, zero-input) solution is representable by

$$x(t) = \phi(t, t_0, x_0, 0) = \Phi(t, t_0)x_0 = \Phi(t - t_0)x_0$$

where $\Phi(t - t_0)$ is called the state transition matrix.

$\Phi(t)$ is called the fundamental matrix.

Note that $\Phi(t)^{-1} = \Phi(-t)$ from the uniqueness of solution.

For the rest of the note, we assume $t_0 = 0$ WLOG.

Suppose $x_0 = e_i$. Then $x(t) = \Phi(t)e_i$ that is the ith column of $\Phi(t)$. Hence, we have the following.

Fact: $\Phi(t)$ is nonsingular and is uniquely determined by

$$\dot{\Phi} = \boldsymbol{A}\Phi, \qquad \Phi(0) = \boldsymbol{I}$$

$$\Downarrow$$

$$\Phi(t) = \boldsymbol{I} + \int_0^t \boldsymbol{A}\Phi(\tau)d\tau.$$

By the Picard iteration (repeated substitution of RHS into the integral for Φ)

$$\Phi(t) = \boldsymbol{I} + t\boldsymbol{A} + \frac{t^2}{2!}\boldsymbol{A}^2 + \cdots =: e^{\boldsymbol{A}t}$$

is the solution.

To this end, the unforced solution is

$$x(t) = e^{\boldsymbol{A}t}x_0$$

$$y(t) = Ce^{At}x_0.$$

Properties of Matrix Exponential e^{At}

1.
$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

2.
$$e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$$

3. e^{At} is nonsingular and
$$[e^{At}]^{-1} = e^{-At}$$

4. For nonsingular P,
$$e^{PAP^{-1}t} = Pe^{At}P^{-1}$$

5.
$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1} = \mathcal{L}^{-1}(Is^{-1} + As^{-2} + A^2s^{-3} + \cdots)$$

Proof: 1) and 4) are obvious from the series representation of e^{At}
2) For all x_0,

$$e^{A(t_1+t_2)}x_0 = x(t_1 + t_2) = e^{At_1}x(t_2) = e^{At_1}e^{At_2}x_0$$

3) 2) $\Rightarrow$ $e^{At}e^{-At} = I \Rightarrow [e^{At}]^{-1} = e^{-At} \Rightarrow$ e^{At} nonsingular.
5) Taking LT's of $\dot{x}(t) = Ax(t)$ with $x(0) = x_0$ and $x(t) = e^{At}x_0$, we get

$$sX(s) = AX(s) + x_0 \quad \Rightarrow \quad X(s) = (sI - A)^{-1}x_0$$

and
$$X(s) = \mathcal{L}e^{At} \cdot x_0,$$
respectively.

$$(sI - A)^{-1} = \mathcal{L}e^{At} = \mathcal{L}\left(\sum_{k=0}^{\infty}\frac{t^kA^k}{k!}\right) = \sum_{k=0}^{\infty}A^ks^{-k-1}.$$

Computation of Matrix Exponential e^{At}

1. Use $e^{PAP^{-1}t} = Pe^{At}P^{-1}$ where PAP^{-1} is the Jordan form of A

2. Use $e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$

3. Use $e^{At} = \mathcal{L}^{-1}(s\boldsymbol{I} - \boldsymbol{A})^{-1}$

4. Use Cayley-Hamilton theorem to find the minimal polynomial $g(\lambda)$ such that $e^{At} = g(\boldsymbol{A})$.

<u>Forced Solution</u>
Fact: the forced solution is

$$x(t) = \phi(t, 0, x_0, u) = \underbrace{e^{At}x_0}_{\text{z. i. sol.}} + \underbrace{\int_0^t e^{A(t-\tau)}\boldsymbol{B}u(\tau)d\tau}_{\text{z. s. sol.}}$$

Proof: At $t = 0$,

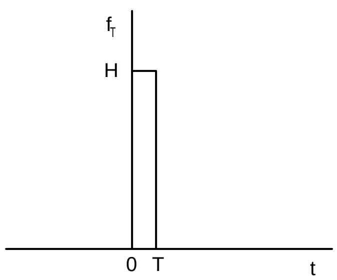
$$\phi(0, 0, x_0, u) = x_0$$

$$\dot{\phi}(t, 0, x_0, u) = \boldsymbol{A}e^{At}x_0 + \boldsymbol{B}u(t) + \int_0^t \boldsymbol{A}e^{A(t-\tau)}\boldsymbol{B}u(\tau)d\tau = \boldsymbol{A}\phi(t, 0, x_0, u) + \boldsymbol{B}u(t).$$

Hence,

$$y(t) = \boldsymbol{C}e^{At}x_0 + \int_0^t \boldsymbol{C}e^{A(t-\tau)}\boldsymbol{B}u(\tau)d\tau + \boldsymbol{D}u(t).$$

<u>Impulse Response and Convolution</u>



Unit impulse (Dirac delta function):

$$\delta(t) = \lim_{T \to 0} f_T(t) \quad \text{with} \quad HT = 1$$

such that for continuous function f,

$$f(t) = \int_0^\infty f(\tau)\delta(t-\tau)d\tau.$$

Impulse Response Matrix ($G(t)$): the ith column of $G(t)$ is $y(t)$ when $x_0 = 0$ and $u = \delta e_i$

$$g_i(t) = G(t)e_i = \begin{cases} Ce^{At}Be_i + De_i\delta(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\Downarrow$$

$$\begin{aligned} y(t) &= Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}B\left[\sum_{i=1}^m u_i(\tau)e_i\right]d\tau + D\left[\sum_{i=1}^m u_i(t)e_i\right] \\ &= Ce^{At}x_0 + (G * u)(t). \end{aligned}$$

Equivalence

Consider the change of coordinate of the state space such that $\bar{x} = Px$. Then

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

$$\Downarrow$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \quad y = \bar{C}\bar{x} + \bar{D}u$$

where

$$\bar{A} = PAP^{-1} \quad \bar{B} = PB \quad \bar{C} = CP^{-1} \quad \bar{D} = D.$$

Hence, two systems represented by $[A, B, C, D]$ and $[\bar{A}, \bar{B}, \bar{C}, \bar{D}]$ are equivalent because the only difference is the coordinate system of the state space.

Modes

If the matrix A has m distinct eigenvalues λ_i with multiplicity p_i, then every element of e^{At} is a linear combination of $t^k e^{\lambda_i t}$, $k = 0, \cdots, p_i$, $i = 1, \cdots, m$ in view of Jordan form of A.

$t^k e^{\lambda_i t}$ is called a mode.

Dynamic characteristics of a system is mainly determined by its modes.

3.2 Input-Output Description

Input-output description: relationship between input and output when $x_0 = 0$.

$x_0 = 0$ is assumed throughout this section.

Input-output description in time domain is given by convolution between input and impulse response matrix:

$$y(t) = (G * u)(t) = \int_0^t G(t-\tau)u(\tau)d\tau = \int_0^\infty G(t-\tau)u(\tau)d\tau$$

In Laplace domain,

$$Y(s) = \int_0^\infty \left(\int_0^\infty G(t-\tau)u(\tau)d\tau \right) e^{-st}dt$$

$$= \int_0^\infty \left(\int_0^\infty G(t-\tau)e^{-s(t-\tau)}dt \right) u(\tau)e^{-s\tau}d\tau = \int_{-\tau}^\infty G(v)e^{-sv}dv \int_0^\infty u(\tau)e^{-s\tau}d\tau$$

$$= \int_0^\infty G(v)e^{-sv}dv \int_0^\infty u(\tau)e^{-s\tau}d\tau = G(s)U(s)$$

$$\Downarrow$$

The transfer function matrix $G(s)$ is the Laplace transform of impulse response matrix.

Taking Laplace transform of state DE and read out map,

$$sX(s) = AX(s) + BU(s)$$

$$Y(s) = CX(s) + DU(s)$$

$$\Downarrow$$

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

$$\Downarrow$$

$$G(s) = C(sI - A)^{-1}B + D = \frac{C adj(sI - A)B + \chi_A(s)D}{\chi_A(s)}$$

where $(sI - A)^{-1}$ is called resolvent matrix and $\chi_A(s) = det(sI - A)$.

$$(sI - A)^{-1} = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1} = \frac{1}{s}\sum_{k=0}^{\infty}\frac{A^k}{s^k} \quad \forall |s| > \rho_A = \lambda_{max}(A)$$

$$\Downarrow$$

$$\lim_{s\to\infty} G(s) = D \quad \text{High Frequency Gain}$$

D: finite $\Rightarrow$ system is proper

$D = 0 \Rightarrow$ system is strictly proper

Suppose $[A, B, C, D]$ and $[\bar{A}, \bar{B}, \bar{C}, \bar{D}]$ are equivalent. Then

$$\bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = CP^{-1}(sI - PAP^{-1})^{-1}PB + D$$

$$= CP^{-1}[P(sI - A)P^{-1}]^{-1}PB + D = C(sI - A)^{-1}B + D.$$

Fact: state space representation of an I/O description is not unique.

Realization problem (Given G, what is the state space realization $[A, B, C, D]$ whose transfer function matrix is G?) will be addressed later in a separate chapter.

Smith-McMillan Form of $G(s)$

Def.: A polynomial matrix $U(s)$ is unimodular if it has inverse that is also polynomial matrix.

Def.: The normal rank of a polynomial matrix $U(s)$ is the maximally possible rank of $U(s)$ for at least one $s \in \mathbf{C}$.

Theorem: $U(s)$ is unimodular iff $detU(s) = const.$ (independent of s).

Proof: ($\Rightarrow$) $U(s)U^{-1}(s) = I$

$$detU(s)detU^{-1}(s) = 1.$$

Both $detU(s)$ and $detU^{-1}(s)$ are polynomial. $\Rightarrow$ They must be constants.

($\Leftarrow$) $detU(s) = c \Rightarrow$

$$U^{-1}(s) = \frac{adjU(s)}{c}$$

$adjU(s)$ is polynomial $\Rightarrow$ so is $U^{-1}(s)$.

Theorem (Smith form): Let $P(s)$ be a polynomial matrix of normal rank r (i.e. of rank r for almost all s). Then through a sequence of elementary row and column operations, $P(s)$ may be transformed into Smith form:

$$S(s) = diag\{\epsilon_1(s), \epsilon_2(s), \cdots, \epsilon_r(s), 0, \cdots, 0\}$$

where each invariant factor $\epsilon_i(s)$ is a monic polynomial satisfying divisibility property that ϵ_i divides ϵ_{i+1} without remainder.

Moreover, if we define the determinantal divisors

$$D_0(s) = 1$$

$D_i(s) =$ GCD of all $i \times i$ minors of $P(s)$ that is normalized to be monic,

then

$$\epsilon_i(s) = \frac{D_i(s)}{D_{i-1}(s)}.$$

Proof: Steps for the reduction to Smith form:

1. bring to the (1,1) position the element of least degree in $P(s)$.
2. $P_{21}(s) = H(s)P_{11}(s) + R(s)$ where $R(s)$ is either zero or such that $deg R(s) < deg P_{11}(s)$. By elementary operation, make $P_{21} = R(s)$.
   (a) If $P_{21}(s) = 0$, proceed to the next step
   (b) If $P_{21}(s) \neq 0$, interchange 1st and 2nd rows and repeat step 2) until $P_{21}(s) = 0$. Since $deg P_{11}(s)$ drops in each cycle, this process terminate in finite time.
3. Similar to Step 2), make all the first column elements zero except $P_{11}(s)$
4. Similar to Steps 2) and 3), make all the first row elements zero except $P_{11}(s)$
5. If Step 4) introduce nonzero first column elements below $P_{11}(s) = 0$, go back to Step 2). Since $deg P_{11}(s)$ drops in each cycle, this process terminate in finite time and give

$$\begin{bmatrix} a(s) & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}$$

6. If any element of columns $2, 3, \cdots$ is not divisible by $a(s)$, we add this column to the first column and then go back to 2). Again this process terminate in finite time and give the above form in which $a(s)$ divides every other nonzero element. Put $\epsilon_1(s) = a(s)$ where $a(s)$ is monic WLOG.

7. Remove first row and first column and repeat the above entire procedure. Then we get

$$\begin{bmatrix} \epsilon_1(s) & 0 & 0 & \cdots & 0 \\ 0 & \epsilon_2(s) & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & * & \cdots & * \end{bmatrix}$$

   where ϵ_2 is divisible by ϵ_1 and every $*$ is divisible by $\epsilon_2(s)$.

8. Repeating the procedure similar to 7), we get Smith form.

$\exists$ r nonzero ϵ_i since the rank is preserved under elementary operation.

The second part of the theorem follows from the Binet-Cauchy theorem (see Kailath, pp. 649). $D_i(s)$ are invariant under elementary operations and thus they are the same for P and S. Hence, we have

$$D_1(s) = \epsilon_1(s)$$

$$D_2(s) = \epsilon_1(s)\epsilon_2(s)$$

$$\vdots$$

$$D_r(s) = \epsilon_1(s)\cdots\epsilon_r(s)$$

Theorem (Smith-McMillan form): Let $G(s)$ be a rational matrix of normal rank r. Then through a sequence of elementary row and column operations, $G(s)$ may be transformed into Smith-McMillan form:

$$M(s) = diag\{\frac{\epsilon_1(s)}{\psi_1(s)}, \frac{\epsilon_2(s)}{\psi_2(s)}, \cdots, \frac{\epsilon_r}{\psi_r(s)}, 0, \cdots, 0\}$$

where monic polynomials $\epsilon_i(s), \psi_i(s)$ are coprime and divisibility property that ϵ_i ($\psi_{i+1}(s)$) divides ϵ_{i+1} ($\psi_i(s)$) without remainder.

Proof: Let $d(s)$ be the least common multiple of all the denominators of the elements of G. Then

$$G(s) = \frac{P(s)}{d(s)}$$

where $\boldsymbol{P}$ is a polynomial matrix. Let

$$S = diag\{\bar{\epsilon}_1(s), \cdots, \bar{\epsilon}_r(s), 0 \cdots 0\}$$

be the smith form of $\boldsymbol{P}$ and let

$$\boldsymbol{M}(s) = \frac{S(s)}{d(s)}$$

where $\frac{\epsilon_i}{\psi_i}$ is obtained by all possible cancellation.

Divisibility properties are obvious from that of $\bar{\epsilon}_i(s)$'s.

Poles and Zeros of G

Let

$$p(s) = \psi_1(s) \cdots \psi_r(s)$$
$$z(s) = \epsilon_1(s) \cdots \epsilon_r(s).$$

Def.: The degree of $p(s)$ is the McMillan degree of $G(s)$.

Def.: Let $G(s)$ be a rational transfer function matrix with Smith-McMillan form. Then the poles and transmission zeros of $G(s)$ are defined to be the roots of p and z, respectively.

Def.: z_0 such that $G(z_0) = 0$ is called the blocking zero of G.

Note: Although $\epsilon_i(s), \psi_i(s)$ are coprime, there may be cancellation of common factor of $z(s)$ and $p(s)$ and thus there may exist a transmission zero that is also a pole.

Fact: Suppose z_0 is not a pole of G. Then z_0 is a transmission zero iff $rankG(z_0) < normalrankG(s)$.

Corollary: Suppose G is square and $det G(s) \not\equiv 0$. Suppose z_0 is not a pole of G. Then z_0 is a transmission zero iff $det G(z_0) = 0$.

Poles and Zeros of $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$

Def.: the eigenvalues of $\boldsymbol{A}$ is called poles of $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$.

Def.: z_0 such that

$$rank \begin{bmatrix} \boldsymbol{A} - z_0 \boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} < normalrank \begin{bmatrix} \boldsymbol{A} - s\boldsymbol{I} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

is called invariant zero of $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}]$.

Fact: Consider constant state feedback $u = \boldsymbol{F}x + v$. The invariant zero is not changed by constant state feedback

Proof:

$$rank \begin{bmatrix} A+BF-z_0I & B \\ C+DF & D \end{bmatrix} = rank \begin{bmatrix} A-z_0I & B \\ CF & D \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix}$$
$$= rank \begin{bmatrix} A-z_0I & B \\ C & D \end{bmatrix}$$

Fact: The invariant zero is not changed under similarity transformation.
Proof:

$$rank \begin{bmatrix} PAP^{-1}-z_0I & PB \\ CP^{-1} & D \end{bmatrix} = rank \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A-z_0I & B \\ C & D \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & I \end{bmatrix}$$
$$= rank \begin{bmatrix} A-z_0I & B \\ C & D \end{bmatrix}$$

Further exploration of invariant zero and its connection with transmission zero will be given in the realisation chapter.

Chapter 4

Stability

4.1 Input-Output Stability

Consider the I/O description of a linear time-invariant system:

$$y(t) = \int_0^t G(t-\tau)u(\tau)d\tau.$$

Def.: a system is bounded input bounded output (BIBO) stable if for $||u||_\infty < \infty$,

$$||y||_\infty < \infty.$$

Def.: a system is L_∞ finite gain stable if $\exists k$ such that for $||u||_\infty < \infty$,

$$||y||_\infty \leq k||u||_\infty.$$

Theorem: TFAE

1. a linear system is BIBO stable.
2. a linear system is L_∞ finite gain stable.
3. $$\int_0^\infty ||Ce^{At}B||dt < \infty$$
   i.e. the impulse response is absolutely integrable.
4. all poles of the system is in LHP.

Proof: $(1 \Rightarrow 2)$ Suppose the contrary. Then $\exists\ u$ such that

$$\frac{||y||_\infty}{||u||_\infty} = \infty.$$

(The proof of this existence is mathematically quite involved and is omitted).

$(2 \Rightarrow 1)$ Obvious.

$(2 \Rightarrow 3)$ Suppose the contrary. Then $\exists\ i, j$ such that

$$\int_0^\infty |G_{ij}(t)|dt = \infty.$$

For all $l \in \mathbf{N}$, $\exists\ t_l$ such that

$$\int_0^{t_l} |G_{ij}(t_l - \tau)|d\tau > l.$$

Define

$$u^l(t) = (0, \cdots, 0, u_j^l(t), 0, \cdots, 0)$$

with

$$u_j^l(t) = \begin{cases} sign[G_{ij}(t_l - t)] & \forall t \in [0, t_l] \\ 0 & \text{elsewhere} \end{cases}.$$

Then,

$$||u^l||_\infty = ||u_j^l||_\infty = 1 \quad \forall\ l$$

and thus

$$y_i^l(t_l) = \int_0^{t_l} G_{ij}(t_l - \tau)u_j^l(\tau)d\tau = \int_0^{t_l} |G_{ij}(t_l - \tau)|d\tau$$

$$\Downarrow$$

$$||y^l||_\infty \geq ||y_i^l||_\infty \geq y_i^l(t_l) > l \quad \forall l.$$

(Contradiction)

$(3 \Rightarrow 2)$ Let $||u||_\infty < \infty$. Then

$$||y||_\infty = \left\| \operatorname*{ess\,sup}_{t \geq 0} \int_0^t G(t-\tau)u(\tau)d\tau \right\|_\infty \leq \operatorname*{ess\,sup}_{t \geq 0} \left\| \int_0^t G(t-\tau)u(\tau)d\tau \right\|_\infty$$

$$\leq \operatorname*{ess\,sup}_{t \geq 0} \int_0^t ||G(t-\tau)u(\tau)||_\infty d\tau \leq \operatorname*{ess\,sup}_{t \geq 0} \int_0^t ||G(t-\tau)||_\infty ||u(\tau)||_\infty d\tau$$

$$\leq \left\{ \operatorname{ess\,sup}_{t\geq 0} \int_0^t \|G(t-\tau)\|_\infty d\tau \right\} \|u\|_\infty = \left\{ \operatorname{ess\,sup}_{t\geq 0} \int_0^t \|G(\tau)\|_\infty d\tau \right\} \|u\|_\infty$$
$$= \left\{ \int_0^\infty \|G(\tau)\|_\infty d\tau \right\} \|u\|_\infty .$$

(3 $\Rightarrow$ 4) Set $H(t) = Ce^{At}B$. Then $G(s) = H(s) + D$. Notice that for all i, j,

$$\int_0^\infty |H_{ij}(t)| dt < \infty .$$

Hence, for all i, j,

$$\sup_{s\in C^+} |H_{ij}(s)| \leq \sup_{s\in C^+} \left| \int_0^\infty H_{ij}(t) e^{-st} dt \right| \leq \sup_{s\in C^+} \int_0^\infty |H_{ij}(t)| \underbrace{|e^{-st}|}_{=|e^{-\sigma t}| \leq 1} dt$$

$$\leq \int_0^\infty |H_{ij}(t)| dt < \infty .$$

(4 $\Rightarrow$ 3) Set $H(t) = Ce^{At}B$.

$$H_{ij}(t) = \sum_{k=1}^{l} \pi_k(t) e^{\lambda_k t}$$

where λ_k is a pole in $\mathbf{C}^-$ and $\pi_k(t)$ is a polynomial in t.

Observe that, for any $\epsilon > 0$, $\exists\ m_k(\epsilon)$ such that

$$|\pi_k(t)| \leq m_k(\epsilon) e^{\epsilon t}.$$

Picking $\mu = -\max_k\{Re\lambda_k\} > 0$, $\epsilon \in (0, \mu)$ and $m(\epsilon) = \sum_{k=1}^{l} m_k(\epsilon)$,

$$|H_{ij}(t)| \leq \sum_{k=1}^{l} |\pi_k(t)| e^{(Re\lambda_k)t} \leq \sum_{k=1}^{l} m_k(\epsilon) e^{\epsilon t} e^{(Re\lambda_k)t}$$

$$\leq \sum_{k=1}^{l} m_k(\epsilon) e^{\epsilon t} e^{-\mu t} = m(\epsilon) e^{-(\mu-\epsilon)t}$$

with $\mu - \epsilon > 0$. Hence, by integrating

$$\int_0^\infty |H_{ij}(t)| dt \leq m(\epsilon)[\mu - \epsilon]^{-1} < \infty .$$

4.2 Lyapunov Stability

Let $\phi(t, x_0, 0)$ be the zero input response of a LTI system.

Def.: The equilibrium point 0 is globally Lyapunov stable if given $\epsilon > 0$, $\exists\, \delta > 0$ such that

- $$||x_0|| < \delta \quad \Rightarrow \quad ||\phi(t, x_0, 0)|| < \epsilon \quad \forall t \geq 0$$
- $\phi(t, x_0, 0)$ is bounded for all $x_0 \in \mathbf{R}^n$.

Def.: The equilibrium point 0 is globally attractive if for all $x_0 \in \mathbf{R}^n$,

$$\lim_{t \to \infty} ||\phi(t, x_0, 0)|| = 0.$$

Def.: The equilibrium point 0 is globally asymptotically stable if it is globally stable and globally attractive.

Def.: The equilibrium point 0 is globally exponentially stable if $\exists r > 0, k \geq 1$ such that

$$||x(t)|| \leq k||x(0)||e^{-rt} \quad \forall t \geq 0$$

for all $x_0 \in \mathbf{R}^n$.

Theorem: 0 is globally Lyapunov stable iff

- all eigenvalues of $\boldsymbol{A}$ have nonpositive real parts
- eigenvalues with zero real parts are distinct root of characteristic polynomial.

Proof: $x(t) = e^{At}x_0 \Rightarrow$

0 is globally Lyapunov stable iff $\exists \boldsymbol{K} > 0\ ||e^{At}|| \leq \boldsymbol{K}$ for all t. Let $\boldsymbol{P}$ be the nonsingular matrix such that $\hat{\boldsymbol{A}} = \boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1}$ is in Jordan form. Since $e^{\hat{A}t} = \boldsymbol{P}e^{\hat{A}t}\boldsymbol{P}^{-1}$, then

$$||e^{\hat{A}t}|| \leq ||\boldsymbol{P}||||e^{At}||||\boldsymbol{P}^{-1}||$$
$$||e^{At}|| \leq ||\boldsymbol{P}^{-1}||||e^{\hat{A}t}||||\boldsymbol{P}||$$
$$\Downarrow$$

$||e^{At}||$ bounded $\Leftrightarrow$ $||e^{\hat{A}t}||$ bounded $\Leftrightarrow$ every entry of $e^{\hat{A}t}$ bounded.

Since $\hat{\boldsymbol{A}}$ is in Jordan form, every entry of $e^{\hat{A}t}$ is of the form $t^k e^{(\alpha_i + j\omega_i)t}$.

If $\alpha_i < 0$, $t^k e^{(\alpha_i + j\omega_i)t}$ is bounded.

If $\alpha_i = 0$, $t^k e^{(\alpha_i + j\omega_i)t}$ is bounded iff $k = 0$.

Theorem: TFAE

1. 0 is globally attractive.

2. 0 is globally asymptotically stable.

3. 0 is globally exponentially stable.

4. all eigenvalues of A have negative real parts.

5. For all PD symmetric matrix Q, the Lyapunov equation

$$A^t P + PA = -Q$$

has a unique PD symmetric solution.

Proof: $(1 \Leftrightarrow 4)$ Clearly 0 is globally attractive iff

$$\lim_{t \to \infty} ||e^{At}|| = 0$$

or

$$\lim_{t \to \infty} ||e^{\hat{A}t}|| = 0.$$

Since every entry of $e^{\hat{A}t}$ is of the form $t^k e^{(\alpha_i + j\omega_i)t}$, $\lim_{t\to\infty} ||e^{\hat{A}t}|| = 0$ iff $\alpha_i < 0$.

$(1 \Leftrightarrow 3)$ Each entry of e^{At} is bounded by an exponentially decaying envelope.

$(1 \Leftrightarrow 2)$ Obvious since $(1 \Leftrightarrow 3)$

$(5 \Rightarrow 4)$ Consider $V(x) = x^t Px$. Then

$$\dot{V}(x) = x^t A^t Px + x^t PAx = -x^t Qx$$

$$\Downarrow$$

$$\frac{\dot{V}}{V} = -\frac{x^t Qx}{x^t Px} \leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} =: -\alpha < 0$$

$$\Downarrow$$

$$V(t) \leq e^{-\alpha t} V(0)$$

$$\Downarrow$$

$$\lim_{t \to \infty} V(t) = 0$$

$$\Downarrow$$

$$\lim_{t\to\infty} x(t) = 0.$$

$(4 \Rightarrow 5)$ (Proof for existence of PD solution) Let

$$\boldsymbol{P} = \int_0^\infty e^{A^* t} Q e^{At} dt.$$

Then

$$\boldsymbol{A}^*\boldsymbol{P}+\boldsymbol{P}\boldsymbol{A} = \int_0^\infty \boldsymbol{A}^* e^{A^* t} Q e^{At} dt + \int_0^\infty e^{A^* t} Q e^{At} \boldsymbol{A} dt = \int_0^\infty \frac{d}{dt}(e^{A^* t} Q e^{At}) dt = -Q.$$

Hence, $\boldsymbol{P}$ is the symmetric solution of the Lyapunov equation. Since e^{At} is nonsingular,

$$x_0^* \boldsymbol{P} x_0 = \int_0^\infty \underbrace{x_0^* e^{A^* t} Q e^{At} x_0}_{>0} dt > 0 \quad \text{if } x_0 \neq 0.$$

$\Rightarrow$ $\boldsymbol{P}$ is PD.

Remark: Suppose $[\boldsymbol{A}, \boldsymbol{B}, C, \boldsymbol{D}]$ is a realisation of $G(s)$. The notice that, if the poles of $[\boldsymbol{A}, \boldsymbol{B}, C, \boldsymbol{D}]$ and $G(s)$ are the same, the global attractivity is equivalent to the BIBO stability of $G(s)$. Later in the realisation section, it will be shown that, if $[\boldsymbol{A}, \boldsymbol{B}, C, \boldsymbol{D}]$ is a minimal realisation of $G(s)$, the poles of $[\boldsymbol{A}, \boldsymbol{B}, C, \boldsymbol{D}]$ and $G(s)$ are the same.

Chapter 5

Controllability and Observerbility

5.1 Controllability

Def.: A state space is controllable if we can reach any state in some time, starting from any other, by applying a suitable input u.

Def.: A state space is reachable if we can reach any state in some time, starting from the origin, by applying a suitable input u.

Def: $\Omega(t) =$ the class of state reachable in the time interval $[0, t]$.

Lemma: $\Omega = \cup_{t>0}\Omega(t)$ is a subspace of $\mathbf{R}^n$ and thus is called controllable subspace.

Proof: Suppose $x_1, x_2 \in \Omega$. Then there exsist t_1, t_2, u_1, u_2 such that

$$x_1 = \int_0^{t_1} e^{A(t_1-\tau)}Bu_1(\tau)d\tau$$

$$x_2 = \int_0^{t_2} e^{A(t_2-\tau)}Bu_2(\tau)d\tau.$$

WLOG, assume $t_1 \le t_2$ and define

$$\tilde{u}_1(t) = \begin{cases} 0 & \text{if } t < t_2 - t_1 \\ u_1(t - t_2 + t_1) & \text{if } t_2 - t_1 \le t \le t_2 \end{cases}.$$

Then

$$x_1 + x_2 = \int_0^{t_1} e^{A(t_1-\tau)}Bu_1(\tau)d\tau + \int_0^{t_2} e^{A(t_2-\tau)}Bu_2(\tau)d\tau$$

$$\underbrace{=}_{\tau'=\tau+t_2-t_1} \int_{t_2-t_1}^{t_2} e^{A(t_2-\tau')} B u_1(\tau' - t_2 + t_1) d\tau' + \int_0^{t_2} e^{A(t_2-\tau)} B u_2(\tau) d\tau$$

$$= \int_0^{t_2} e^{A(t_2-\tau)} B(\tilde{u}_1 + u_2)(\tau) d\tau \in \Omega.$$

Hence, the addition is well defined. Clearly, the scalar multiplication is also well defined. Moreover, it is clear that all the necessary algebraic laws are satisfied. Hence, the lemma follows.

Fact: TFAE

1. the system is reachable.

2. $$\mathbf{R}^n = \cup_{t>0} \Omega(t).$$

3. $$\langle x, z \rangle = 0 \ \ \forall z \in \Omega(t), \ \forall t > 0 \quad \Rightarrow \quad x = 0.$$

Proof: $(1 \Leftrightarrow 2)$ Obvious

$(2 \Rightarrow 3)$ Obvious

$(3 \Rightarrow 2)$ Suppose the contrary. Then from the lemma, Ω is a subspace of $\mathbf{R}^n$ whose dimension is less than n. Then for any nonzero $x \in \Omega^\perp$,

$$\langle x, z \rangle = 0 \ \ \forall z \in \Omega(t), \ \forall t > 0.$$

(Contradiction!)

If $z \in \Omega(t)$, $\exists u$ such that

$$z = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau.$$

Let $s = t - \tau$ and $\tilde{u}(s) = u(t-s)$. Then

$$\langle x, z \rangle = \left\langle x, \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \right\rangle = \left\langle x, \int_0^t e^{As} B \tilde{u}(s) ds \right\rangle = \int_0^t \left\langle B^* e^{A^* s} x, \tilde{u}(s) \right\rangle ds.$$

Now

$$\langle x, z \rangle = 0 \ \ \forall z \in \Omega(t)$$

$$\Updownarrow$$

$$\int_0^t \left\langle B^* e^{A^* s} x, \tilde{u}(s) \right\rangle ds = 0 \quad \forall \tilde{u}$$

$$\text{Pick } \tilde{u}(s) = B^* e^{A^* s} x \Downarrow \quad \Uparrow B^* e^{A^* s} x = 0$$

$$\int_0^t \left\langle B^* e^{A^* s} x, B^* e^{A^* s} x \right\rangle ds = 0$$

$$\Updownarrow$$

$$B^* e^{A^* s} x = 0 \quad 0 \le s \le t$$

$$\text{Obvious} \Downarrow \quad \Uparrow x^* e^{As} BB^* e^{A^* s} x = 0$$

$$e^{As} BB^* e^{A^* s} x = 0 \quad 0 \le s \le t$$

$$\Updownarrow$$

$$\left(\int_0^t e^{As} BB^* e^{A^* s} ds \right) x = 0.$$

Define the controllability grammian as

$$W_c(t) = \int_0^t e^{As} BB^* e^{A^* s} ds.$$

Then from the above fact, we have the following theorem.

Theorem: TFAE

1. The state space is reachable.
2. $W_c(t)x = 0$ for all $t > 0$ implies $x = 0$.
3. $W_c(t)$ is nonsingular for all $t > 0$.
4. $W_c(t)$ is positive definite for all $t > 0$.
5. $B^* e^{A^* s} x = 0$ for all $s > 0$ implies $x = 0$.

The proof of this theorem is obvious.

Lemma:

$$\Omega(t) = \mathcal{R}(W_c(t)) = \mathcal{R}(L_c L_c^*) = \mathcal{R}(L_c) = \cup_{t>0} \Omega(t)$$

where L_c is the controllability matrix:

$$L_c = [B \; AB \; \cdots \; A^{n-1}B].$$

Proof: The third equality is obvious from the fact established in Section 2.3.

Now

$$W_c(t)x = \int_0^t e^{As} B\tilde{u}(s)ds$$

where

$$\tilde{u}(s) = B^* e^{A^* s} x$$

$$\Downarrow$$

$$\mathcal{R}(W_c(t)) \subset \Omega(t).$$

By Cayley-Hamilton theorem,

$$e^{At} = \sum_{k=0}^{n-1} \phi_k(t) A^k$$

$$\Downarrow$$

$$\int_0^t e^{As} Bu(s)ds = \sum_{k=0}^{n-1} A^k B \left(\int_0^t \phi_k(s)u(s)ds \right) = \sum_{k=0}^{n-1} A^k B u_k(t)$$

$$= [B \; AB \; \cdots \; A^{n-1}B] \begin{bmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_{n-1}(t) \end{bmatrix} = L_c \begin{bmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_{n-1}(t) \end{bmatrix}$$

where

$$u_k(t) = \int_0^t \phi_k(s)u(s)ds.$$

Clearly,

$$\Omega(t) \subset \mathcal{R}(L_c) = \mathcal{R}(L_c L_c^*).$$

Now we have

$$L_c L_c^* = \sum_{k=0}^{n-1} A^k BB^* A^{*k}.$$

On the other hand,

$$W_c(t) = \int_0^t e^{As} BB^* e^{A^* s} ds = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} r_{jk}(t) A^j BB^* A^{*k}$$

where

$$r_{jk}(t) = \int_0^t \phi_j(s)\phi_k(s)ds.$$

Claim: $\mathcal{N}(L_c L_c^*) = \mathcal{N}(W_c(t))$.

Suppose

$$W_c(t)x = 0$$

$$\Downarrow$$

$$B^* e^{A^* s} x = 0, \quad 0 \le s \le t$$

$$\Downarrow$$

$$0 = \frac{d^k}{ds^k}(B^* e^{A^* s} x)\bigg|_{s=0} = B^* A^{*k} x, \quad \forall k$$

$$\Downarrow$$

$$L_c L_c^* x = 0.$$

Conversely suppose

$$L_c L_c^* x = 0$$

$$\Downarrow$$

$$0 = \langle L_c L_c^* x, x \rangle = \sum_{k=0}^{n-1} \langle A^k B B^* A^{*k} x, x \rangle = \sum_{k=0}^{n-1} \|B^* A^{*k} x\|^2$$

$$\Downarrow$$

$$B^* A^{*k} x = 0, \quad k = 0, \cdots, n-1$$

$$\Downarrow$$

$$0 = \sum_{k=0}^{n-1} \phi_k(s) B^* A^{*k} x = B^* \left(\sum_{k=0}^{n-1} \phi_k(s) A^{*k} \right) x = B^* e^{A^* s} x$$

$$\Downarrow$$

$$W_c(t)x = 0.$$

and the claim follows.

For a Hermitian matrix H, $\mathcal{R}(H) = \mathcal{N}(H)^\perp$. Hence, $\mathcal{R}(W_c(t)) = \mathcal{R}(L_c L_c^*)$. To this end, the first three equalities follow.

The last equality follows from the fact that L_c doesn't depend on t.

Theorem: TFAE

1. The state space is controllable.
2. The state space is reachable.
3. $W_c(t)x = 0$ for all $t > 0$ implies $x = 0$.
4. $W_c(t)$ is nonsingular for all $t > 0$.
5. $W_c(t)$ is positive definite for all $t > 0$.
6. $B^* e^{A^* s} x = 0$ for all $s > 0$ implies $x = 0$.
7. L_c has rank n.
8. The matrix $[A - \lambda I, B]$ has full row rank for all $\lambda \in \mathbf{C}$.
9. Let λ and w be any eigenvalue and any corresponding left eigenvector of A, i.e., $w^* A = w^* \lambda$, then $x^* B \neq 0$.

Proof: $(2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 6)$ is proven in the previous theorem.
$(1 \Rightarrow 2)$ Obvious
$(2 \Rightarrow 1)$ For controllability, find u such that

$$x_1 = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau.$$

Let

$$z(t) = x_1 - e^{At}x_0.$$

Seek u in the form:

$$u(\tau) = B^* e^{A^*(t-\tau)} x(t)$$

where $x(t)$ is to be determined. By reachability, $W_c(t)$ is nonsingular and thus pick

$$x(t) = W_c(t)^{-1} z(t).$$

Then

$$e^{At}x_0 + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau = e^{At}x_0 + \int_0^t e^{A(t-\tau)} B B^* e^{A^*(t-\tau)} d\tau x(t)$$
$$= e^{At}x_0 + W_c(t) W_c(t)^{-1} z(t) = x_1.$$

(2 ⇔ 7) Obvious from the lemma

(7 ⇒ 8) Suppose the contrary. Then ∃ nonzero $w \in \mathbf{C}^n$ such that

$$w^*[\boldsymbol{A} - \lambda \boldsymbol{I}, \boldsymbol{B}] = 0$$

$$\Downarrow$$

$$w^* \boldsymbol{A} = \lambda w^* \quad \text{and} \quad w^* \boldsymbol{B} = 0$$

$$\Downarrow$$

$$w^*[\boldsymbol{B}\ \boldsymbol{AB} \cdots \boldsymbol{A}^{n-1}\boldsymbol{B}] = [w^*\boldsymbol{B}\ \lambda w^*\boldsymbol{B} \cdots \lambda^{n-1} w^* \boldsymbol{B}] = 0$$

$$\Downarrow$$

$\boldsymbol{L}_c$ does not have full row rank. (contradiction!)

(8 ⇒ 9) Obvious from the proof of (7 ⇒ 8)

(9 ⇒ 7) Suppose $rank \boldsymbol{L}_c = k < n$. Later in section 5.4, we will show that ∃ nonsingular $\boldsymbol{T}$ such that

$$\boldsymbol{TAT}^{-1} = \begin{bmatrix} \boldsymbol{A}_c & \boldsymbol{A}_{12} \\ 0 & \bar{\boldsymbol{A}}_c \end{bmatrix}, \quad \boldsymbol{TB} = \begin{bmatrix} \boldsymbol{B}_c \\ 0 \end{bmatrix}$$

with $\bar{\boldsymbol{A}}_c \in \mathrm{R}^{(n-k)\times(n-k)}$. Let λ_1 and w_c^* be any eigenvalue and left eigenvector of $\bar{\boldsymbol{A}}_c$, i.e., $w_c^* \bar{\boldsymbol{A}}_c = \lambda_1 w_c^*$. Define $w^* = [0\ w_c^*]$. Then

$$w^*(\boldsymbol{TB}) = 0 \quad \text{and} \quad w^* \boldsymbol{TAT}^{-1} = [0\ \lambda_1 w_c^*] = \lambda_1 w^*$$

$$\Downarrow$$

$$(w^*\boldsymbol{T})\boldsymbol{B} = 0 \quad \text{and} \quad (w^*\boldsymbol{T})\boldsymbol{A} = \lambda_1 (w^* \boldsymbol{T}).$$

(Contradiction!)

In the above theorem, the conditions 8 and 9 are called Popov-Belevitch-Hautus (PBH) tests.

Corollary: Suppose $\boldsymbol{A}$ is stable. Then $(\boldsymbol{A}, \boldsymbol{B})$ is controllable iff the solution to the Lyapunov equation:

$$\boldsymbol{AW}_c + \boldsymbol{W}_c \boldsymbol{A}^* + \boldsymbol{BB}^* = 0$$

is PD.

Proof: Similar to stability theorem, the solution to the Lyapunov equation is the aforementioned controllability grammian and the corollary follows.

Fact: Controllability is preserved under similarity transformation.

Proof: Let $\tilde{A} = TAT^{-1}$ and $\tilde{B} = TB$. Then

$$[\tilde{B}, \cdots, \tilde{A}^{n-1}\tilde{B}] = T[B, \cdots, A^{n-1}B]$$

and the fact follows.

5.2 Observability

Question: given $(u(t), y(t))$, can we deduce what the state was at $t = 0$?

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)$$

$$\Downarrow$$

$$Ce^{At}x(0) = y(t) - \int_0^t Ce^{A(t-s)}Bu(s)ds - Du(t) := v(t).$$

Question: given $v(t)$, can we find the unique solution x of

$$Ce^{At}x = v(t).$$

Notice that given v, $\exists$ at least one x such that $Ce^{At}x = v(t)$. Hence, the problem is uniqueness.

Def.: A state space is observable if the initial state $x(0)$ can be uniquely determined from $(u(t), y(t))$, $t \in [0, t_1]$.

Theorem: the state space is observable iff for any $t > 0$,

$$Ce^{As}x = 0$$

implies that $x = 0$.

Proof: $(\Rightarrow)$ Suppose $\exists\ z \neq 0$ such that

$$Ce^{As}z = 0, \quad 0 \leq s \leq t.$$

Then

$$v(t) = Ce^{At}x = Ce^{At}(x + z), \quad 0 \leq s \leq t.$$

Hence, the initial state that matches (u, y) pair is not unique and thus unobservable.

($\Leftarrow$) Suppose $x_0 \neq \bar{x}_0$, $Ce^{At}x_0 = v(t)$ and $Ce^{At}\bar{x}_0 = v(t)$. Then

$$Ce^{At}(x_0 - \bar{x}_0) = 0$$

where $x_0 - \bar{x}_0 \neq 0$ (contradiction).

Theorem: TFAE

1. the state space is observable.
2. the observability grammian

   $$W_o(t) = \int_0^t e^{A^* s} C^* C e^{As} ds$$

   is nonsingular for some $t > 0$.
3. $W_o(t)$ is positive definite for some $t > 0$.

Proof: ($1 \Rightarrow 2$) Suppose the contrary. Suppose for some $t_0 > 0$.

$$W_o(t_0)(x) = 0, \quad x \neq 0.$$

Then

$$0 = \langle W_o(t_0)x, x\rangle = \int_0^t \langle e^{A^* s} C^* C e^{As} x, x\rangle ds = \int_0^t \|Ce^{As}x\|^2 ds.$$

Hence,

$$Ce^{As}x = 0, \quad 0 \leq s \leq t_0$$

that contradicts observability.

($2 \Leftarrow 1$) Suppose the contrary. Then for some $T > 0$,

$$Ce^{As}x = 0, \quad 0 \leq s \leq T, \quad x \neq 0$$

$$\Downarrow$$

$$\frac{d^k}{ds^k}(Ce^{As}x)\bigg|_{s=0} = CA^k x = 0$$

$$\Downarrow$$

$$Ce^{As}x = 0, \quad s \geq 0.$$

Multipying $e^{A^* s}C^*$ and integrating,

$$\int_0^t e^{A^* s}C^* C e^{As} x ds = 0, \quad \forall t.$$

(Contradiction!)

(2 $\Leftrightarrow$ 3) Obvious

Corollary: Suppose A is stable. Then (C, A) is observable iff the solution to the Lyapunov equation:

$$A^* W_o + W_o A + C^* C = 0$$

is PD.

Theorem: TFAE

1. state space is observable.

2. the observability grammian

   $$W_o(t) = \int_0^t e^{A^* s}C^* C e^{As} ds$$

   is nonsingular for some $t > 0$.

3. $W_o(t)$ is positive definite for some $t > 0$.

4. observability matrix

   $$L_o := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

   has full rank.

5. The matrix $\begin{bmatrix} A - \lambda I \\ B \end{bmatrix}$ has full column rank for all $\lambda \in \mathbf{C}$.

6. Let λ and w be any eigenvalue and any corresponding right eigenvector of A, i.e., $Ax = \lambda x$, then $Cx \neq 0$.

Proof: $(1 \Rightarrow 2 \Rightarrow 3)$ is proven in the previous theorem.

$(4 \Rightarrow 1)$ Suppose the contrary. Then for some $t > 0$,

$$Ce^{As}x = 0, \quad 0 \le s \le t, \quad x \ne 0$$

$$\Downarrow$$

$$\frac{d^k}{ds^k}(Ce^{As}x)\bigg|_{s=0} = 0 = CA^k x$$

$$\Downarrow$$

$$0 = Cx = CAx = \cdots = CA^{n-1}x, \quad x \ne 0$$

$$\Downarrow$$

L_o does not have full rank. (contradiction!)

$(1 \Rightarrow 4)$ Suppose the contrary. Then

$$0 = Cx = CAx = \cdots = CA^{n-1}x, \quad x \ne 0.$$

By Cayley-Hamilton Theorem,

$$CA^n x = 0$$

$$\Downarrow$$

$$Ce^{As}x = 0, \quad 0 \le s \le T.$$

(contradiction!)

$(4 \Rightarrow 5 \Rightarrow 6)$ Obvious from the duality in the next section.

In the above fact, the conditions 5 and 6 are called Popov-Belevitch-Hautus (PBH) tests.

Fact: observability is preserved under similarity transformation.

Proof: Let $\tilde{A} = TAT^{-1}$ and $\tilde{C} = CT^{-1}$. Then

$$\begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T^{-1}$$

and the fact follows.

5.3 Duality

Fact: (C, A) observable iff (A^*, C^*) controllable.

Proof:

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full rank

$$\Updownarrow$$

$$[C^* \; A^*C^* \; \cdots \; {A^*}^{n-1}C^*]$$

has full rank.

5.4 Kalman Decomposition

Theorem: Suppose L_c has rank n_1. Then $\exists \; \bar{x} = Px$ where P nonsingular such that

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = [\bar{C}_c \; \bar{C}_{\bar{c}}] \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Du,$$

and

$$\dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u$$

$$\bar{y} = \bar{C}_c \bar{x}_c + Du$$

is controllable and the transfer matrices of two systems are the same.

Proof: Let $q_1, \cdots, q_{n_1}$ be any n_1 linearly independent columns of L_c. Define

$$P^{-1} = Q = [q_1 \; \cdots \; q_{n_1} \; q_{n_1+1} \; \cdots \; q_n]$$

where q_i, $i = n_1 + 1, \cdots, n$ are chosen such that Q is nonsingular.

Let $\bar{x} = Px$. Then for $i = 1, \cdots, n$, the ith column of $\bar{A} = PAP^{-1} = P[Aq_1 \; \cdots \; Aq_n]$ is the representation of Aq_i w.r.t. q_i. However, any columns of L_c linearly depend on $\{q_i\}_{i=1}^{n_1}$ and, by Cayley-Hamilton theorem, any columns of A^kB for all k linearly depend on $\{q_i\}_{i=1}^{n_1}$. To this end, $\{Aq_i\}_{i=1}^{n_1}$

can be written as a linear combination of $\{q_i\}_{i=1}^{n_1}$ and thus doesn't depend on $\{q_i\}_{i=n_1+1}^{n}$.

$\Rightarrow$ $\bar{A}$ has the desired form.

The columns of $\bar{B} = PB$ are the representation of those of B w.r.t. $\{q_i\}_{i=1}^{n}$.

Since B is part of L_c, columns of B can be written as a linear combination of $\{q_i\}_{i=1}^{n_1}$ and thus doesn't depend on $\{q_i\}_{i=n_1+1}^{n}$.

$\Rightarrow$ $\bar{B}$ has the desired form.

Let $\bar{L}_c$ be the controllability matrix of $(\bar{A}, \bar{B})$. Clearly, L_c and $\bar{L}_c$ have the rank n_1.

Note that

$$\bar{L}_c = \begin{bmatrix} \bar{B}_c & \bar{A}_c\bar{B}_c & \cdots & \bar{A}_c^{n-1}\bar{B}_c \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \bar{L}_{cc} & \bar{A}_c^{n_1}\bar{B}_c & \cdots & \bar{A}_c^{n-1}\bar{B}_c \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

where $\bar{L}_{cc}$ is the controllability matrix of $(\bar{A}_c, \bar{B}_c)$. Since the Cayley-Hamilton theorem dictates that the columns of $\bar{A}_c^k\bar{B}_c$ with $k \geq n_1$ are linearly dependent on the columns of $\bar{L}_{cc}$, $rank\bar{L}_c = n_1$ implies $rank\bar{L}_{cc} = n_1$. Hence, the reduced system is controllable.

Notice that

$$\begin{bmatrix} sI - \bar{A}_c & -\bar{A}_{12} \\ 0 & sI - \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} (sI - \bar{A}_c)^{-1} & (sI - \bar{A}_c)^{-1}\bar{A}_{12}(sI - \bar{A}_{\bar{c}})^{-1} \\ 0 & (sI - \bar{A}_{\bar{c}})^{-1} \end{bmatrix} = I$$

$$\Downarrow$$

$$\begin{aligned} C(sI - A)^{-1}B + D &= [\bar{C}_c\ \bar{C}_{\bar{c}}] \begin{bmatrix} sI - \bar{A}_c & -\bar{A}_{12} \\ 0 & sI - \bar{A}_{\bar{c}} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + D \\ &= [\bar{C}_c\ \bar{C}_{\bar{c}}] \begin{bmatrix} (sI - \bar{A}_c)^{-1} & (sI - \bar{A}_c)^{-1}\bar{A}_{12}(sI - \bar{A}_{\bar{c}})^{-1} \\ 0 & (sI - \bar{A}_{\bar{c}})^{-1} \end{bmatrix} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + D \\ &= \bar{C}_c(sI - \bar{A}_c)^{-1}\bar{B}_c + D. \end{aligned}$$

The state space $\bar{x}$ is partitioned into two orthogonal subspaces

$$\begin{bmatrix} \bar{x}_c \\ 0 \end{bmatrix} \quad \text{controllable}$$

and

$$\begin{bmatrix} 0 \\ \bar{x}_{\bar{c}} \end{bmatrix} \quad \text{uncontrollable (not affected by input).}$$

Since

$$x = P^{-1}\bar{x} = [q_1 \ \cdots \ q_{n_1} \ q_{n_1+1} \ \cdots \ q_n] \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix},$$

the controllable subspace is the span of $\{q_i\}_{i=1}^{n_1}$ ($\mathcal{R}(L_c)$) and the uncontrollable subspace is its orthogonal complement.

Theorem: Suppose L_o has rank n_2. Then $\exists\ \bar{x} = Px$ where P nonsingular such that

$$\begin{bmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u$$

$$y = [\bar{C}_o \ 0] \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + Du$$

and

$$\dot{\bar{z}}_o = \bar{A}_o \bar{z}_o + \bar{B}_o u$$
$$\bar{y} = \bar{C}_o \bar{z}_o + Du$$

is observable and the transfer matrices of two systems are the same.

Proof: the theorem follows from the duality.

Observable and unobservable subspaces are defined similarly.

Kalman Decomposition Theorem: A linear system can be equivalently transformed into

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

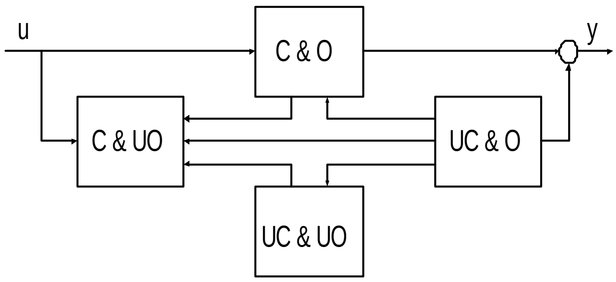
$$y = [\bar{C}_{co} \ 0 \ \bar{C}_{\bar{c}o} \ 0] \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + Du$$

where $\bar{x}_{co}$ is controllable and observable, $\bar{x}_{c\bar{o}}$ is controllable but unobservable, $\bar{x}_{\bar{c}o}$ is observable but uncontrollable, and $\bar{x}_{\bar{c}\bar{o}}$ is uncontrollable and unonservable. Furthermore, the transfer function matrix is

$$\bar{C}_{co}(sI - \bar{A}_{co})^{-1}\bar{B}_{co} + D$$

which depends only on the controllable and observable part.

Proof: Combine previous two theorems successively.



Remark: Even if the transfer function matrices of full system and controllable and observable part are the same, the responses with nonzero initial condition are completely different.

5.5 Hidden Modes

Let λ be an eigenvalue of $\boldsymbol{A}$.

Def.: There is an uncontrollable hidden mode at λ if there is a generalized eigenvector associated λ contained in the uncontrollable subspace.

Def.: There is an unobservable hidden mode at λ if there is a generalized eigenvector associated λ contained in the unobservable subspace.

Def.: There is a hidden mode at λ if there is an uncontrollable or an unobservable hidden mode at λ.

Theorem:

1. There is an uncontrollable hidden mode at λ iff $rank[\boldsymbol{A} - \lambda \boldsymbol{I}\ \boldsymbol{B}] < n$.
2. There is an unobservable hidden mode at λ iff $rank \begin{bmatrix} \boldsymbol{A} - \lambda \boldsymbol{I} \\ \boldsymbol{C} \end{bmatrix} < n$.

Proofs are similar to those in Controllability and Observability section.

5.6 Stabilizability and Detectability

Def.: A system is stabilizable if $\exists$ $u = Fx$ such that $A + BF$ is stable.

Def.: A system is detectable if $A + LC$ is stable for some L.

Theorem: TFAE

1. (A, B) is stabilizable.
2. there are no unstable uncontrollable hidden mode.
3. $\bar{A}_{\bar{c}}$ in the controllability decomposition must be stable.
4. $[A - \lambda I, B]$ has full row rank for all $Re\lambda \geq 0$.
5. For all λ and x such that $x^* A = x^* \lambda$ and $Re\lambda \geq 0$, $x^* B \neq 0$.

Proof is similar to that in Controllability section.

Theorem: TFAE

1. (C, A) is detectable.
2. there are no unstable unobservable hidden mode.
3. $\bar{A}_{\bar{o}}$ in the observability decomposition must be stable.
4. $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank for all $Re\lambda \geq 0$.
5. For all λ and x such that $Ax = \lambda x$ and $Re\lambda \geq 0$, $Cx \neq 0$.

Proof is similar to that in Observability section.

Def (modal controllability): The mode λ is controllable if $x^* B \neq 0$ for all left eigenvectors x associated with λ.

Def (modal observability): The mode λ is observable if $Cx \neq 0$ for all right eigenvectors x associated with λ.

Fact:

- a system is controllable (observable) iff every mode is controllable (observable).
- a system is stabilizable (detectable) iff every unstable mode is controllable (observable).

Chapter 6

Realization Theory

We have shown how one can get G from $[A, B, C, D]$.

Question: Given G, what is the state space realization $[A, B, C, D]$ whose transfer function matrix is G?

Clearly, the realization is not unique since one can add some uncontrollable and unobservable state equations.

6.1 Minimal Realization

Def.: The realization of G is $R = [A, B, C, D]$ such that its transfer function matrix is G.

Def.: The realization $R = [A, B, C, D]$ of G is minimal (irreducible) if its dimension is minimal among all realizations of G.

Fact: TFAE

- $R = [A, B, C, D]$ and $\bar{R} = [\bar{A}, \bar{B}, \bar{C}, \bar{D}]$ are realizations of G.
- R and $\bar{R}$ have the same impulse response.
- $$CA^iB = \bar{C}\bar{A}^i\bar{B} \quad i = 0, 1, \cdots \quad \text{and} \quad D = \bar{D}.$$

Let $R = [A, B, C, D]$ be a realization of G.

Consider the impulse response

$$G(t) - D\delta(t) = Ce^{At}B = \sum_{i=0}^{\infty} \frac{CA^iB}{i!} t^i.$$

Taking Laplace tranform, we obtain an expansion at ∞:

$$G(s) = G(\infty) + \sum_{i=0}^{\infty} G_i s^{-(i+1)} \quad |s| > \max\{|\lambda| : \lambda \in \boldsymbol{P}[G(s)]\}$$

where the Markov parameters $G_i = CA^iB$.

Hankel matrices $\boldsymbol{H}_l$ of order l

$$\boldsymbol{H}_l = [G_{i+j}]_{i,j=0}^{l} = [CA^{i+j}B]_{i,j=0}^{l} = \boldsymbol{L}_{ol}\boldsymbol{L}_{cl}$$

where

$$\boldsymbol{L}_{ol} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^l \end{bmatrix} \quad \boldsymbol{L}_{cl} = [B\ AB\ \cdots\ A^lB].$$

Fact: For any realisation $\boldsymbol{R} = [A, B, C, D]$ of G,

1.
$$rank\boldsymbol{H}_l = rank\boldsymbol{H}_{n-1} \quad \forall l \geq n-1.$$

2.
$$rank\boldsymbol{H}_{n-1} \text{ is independent of the given realisation.}$$

Proof: 1) Let $l \geq n-1$. Let

$$\tilde{\boldsymbol{L}}_{o,n-1} = \begin{bmatrix} \boldsymbol{L}_{o,n-1} \\ 0 \end{bmatrix}, \quad \tilde{\boldsymbol{L}}_{c,n-1} = \begin{bmatrix} \boldsymbol{L}_{c,n-1} & 0 \end{bmatrix}, \quad \tilde{\boldsymbol{H}}_{n-1} = \begin{bmatrix} \boldsymbol{H}_{n-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

By Cayley-Hamilton theorem, $\exists$ nonsingular matrices $\boldsymbol{L}$ and $\boldsymbol{R}$ such that

$$\boldsymbol{L}\boldsymbol{L}_{o,l} = \tilde{\boldsymbol{L}}_{o,n-1} \quad \text{and} \quad \boldsymbol{L}_{c,l}\boldsymbol{R} = \tilde{\boldsymbol{L}}_{c,n-1}$$

$$\Downarrow$$

$$\boldsymbol{L}\boldsymbol{H}_l\boldsymbol{R} = \tilde{\boldsymbol{L}}_{o,n-1}\tilde{\boldsymbol{L}}_{c,n-1} = \tilde{\boldsymbol{H}}_{n-1}$$

$$\Downarrow$$

$$rank\boldsymbol{H}_l = rank\tilde{\boldsymbol{H}}_{n-1} = rank\boldsymbol{H}_{n-1}.$$

2) Let $\tilde{\boldsymbol{R}} = [\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$ with dimension $\tilde{n}$ be another realisation of G.

Let $l \geq \max\{n-1, \tilde{n}-1\}$ and notice that

$$H_l = [CA^{i+j}B]_{i,j=0}^{l} = [\tilde{C}\tilde{A}^{i+j}\tilde{B}]_{i,j=0}^{l} = \tilde{H}_l.$$

By 1),

$$rank H_{n-1} = rank H_l = rank \tilde{H}_l = rank \tilde{H}_{\tilde{n}-1}.$$

Theorem:

1.
$$rank H_l \geq rank H_{l-1} \quad \forall l \in \mathrm{N}.$$

2.
$$\max\{rank H_l; l \in \mathrm{N}\} = rank H_{n-1} = rank\{[CA^{i+j}B]_{i,j=0}^{n-1}\}.$$

Proofs are obvious.
Theorem: Let $\delta_M = rank[L_o L_c] = rank H_{n-1}$.

1.
$$n \geq \delta_M.$$

2. $n = \delta_M \Leftrightarrow (A, B)$ is controllable and (C, A) is observable.

Proof: By Sylvester inequality,

$$rank L_o + rank L_c - n \leq \delta_M \leq \min\{rank L_o, rank L_c\} \leq n.$$

Notice that

$$(C, A) \text{ is observable} \quad \Leftrightarrow \quad rank L_o = n$$
$$(A, B) \text{ is controllable} \quad \Leftrightarrow \quad rank L_c = n.$$

Hence, 1) and 2) follow.
Corollary:

- δ_M is the minimal dimension of any realisation.
- a realisation is minimal iff (A, B) is controllable and (C, A) is observable.

Fact: δ_M is equal to the McMillan degree of G.

Theorem: Let $\boldsymbol{R}$ be a minimal realisation of G and $\tilde{\boldsymbol{R}}$ be another realisation. Then $\tilde{\boldsymbol{R}}$ is minimal iff $\boldsymbol{R}$ and $\tilde{\boldsymbol{R}}$ are equivalent. Furthermore, the equivalence transform is $\tilde{x} = Tx$ where

$$T = (\tilde{L}_o^* \tilde{L}_o)^{-1} \tilde{L}_o^* L_o$$

$$T^{-1} = L_c \tilde{L}_c^* (\tilde{L}_c \tilde{L}_c^*)^{-1}.$$

Proof: $\Leftarrow$) Obvious

$\Rightarrow$) dimension of $\boldsymbol{R}$ and $\tilde{\boldsymbol{R}}$ is n. Hence,

$$D = G(\infty) = \tilde{D}$$

$$L_o L_c = H_{n-1} = \tilde{H}_{n-1} = \tilde{L}_o \tilde{L}_c \quad (*)$$

$$L_o A L_c = \tilde{L}_o \tilde{A} \tilde{L}_c$$

where

$$rank L_o = rank \tilde{L}_o = rank L_c = rank \tilde{L}_c = n.$$

Hence $\tilde{L}_o^* \tilde{L}_o$ and $\tilde{L}_c \tilde{L}_c^*$ are positive definite and, thus, nonsingular. Therefore

$$T_1 = (\tilde{L}_o^* \tilde{L}_o)^{-1} \tilde{L}_o^* L_o \quad \text{and} \quad T_2 = L_c \tilde{L}_c^* (\tilde{L}_c \tilde{L}_c^*)^{-1}$$

are well defined. From $(*)$, $T_1 T_2 = I$. Let $T = T_1$. Now

$$T L_c = (\tilde{L}_o^* \tilde{L}_o)^{-1} \tilde{L}_o^* L_o L_c = (\tilde{L}_o^* \tilde{L}_o)^{-1} \tilde{L}_o^* \tilde{L}_o \tilde{L}_c = \tilde{L}_c$$

and

$$L_o T^{-1} = L_o L_c \tilde{L}_c^* (\tilde{L}_c \tilde{L}_c^*)^{-1} = \tilde{L}_o \tilde{L}_c \tilde{L}_c^* (\tilde{L}_c \tilde{L}_c^*)^{-1} = \tilde{L}_o.$$

Then

$$\tilde{B} = TB \quad \text{and} \quad \tilde{C} = CT^{-1}$$

$$\tilde{A} = TAT^{-1}.$$

6.2 Controllable Canonical Form

Assumptions:

1. (A, B) is controllable.

2. WLOG (by removing null space), B has full column rank.

Selection of basis

Let b_i denote the ith column of B. Since $L_c = [B \cdots A^{n-1}B]$ has rank n, $\exists$ n linearly independent columns of the form $A^j b_i$. Choose first n linearly independent columns including all b_i's.

Lemma: If $A^j b_i$ is linearly dependent on previously selected columns, then so are all columns $A^m b_i$, $m \geq j$.

Proof: Let

$$A^j b_i = \sum \alpha_l a_l$$

where a_l are selected columns before $A^j b_i$. Then

$$A^m b_i = \sum \alpha_l A^{m-j} a_l$$

where all $A^{m-j} a_l$'s precede $A^m b_i$ and thus are linearly dependent on selected columns before $A^m b_i$.

Hence, for all $i \in [1, m]$, we find a least integer $k_i \in [1, n]$, called ith controllability indices, such that $A^{k_i} b_i$ is linearly dependent on previously selected columns. Now select the m families

$$\{A^j b_i\}_{j=0}^{k_i - 1}.$$

Clearly,

$$\sum_{i=1}^{m} k_i = n.$$

Then for all $l \in [1, m]$, $\exists$ γ_j^{il}'s such that

$$A^{k_l} b_l = -\sum_{j=0}^{k_l} \sum_{i=1}^{m} \gamma_j^{il} A^j b_i$$

where

$$\gamma_{k_l}^{ll} = \gamma_{k_l}^{l+1,l} = \cdots = \gamma_{k_l}^{ml} = 0, \qquad (\dagger)$$

and for all $i \in [1, m]$ such that $k_i \leq k_l$

$$\gamma^{il}_{k_i} = \gamma^{il}_{k_i+1} = \cdots = \gamma^{il}_{k_l} = 0. \quad (\ddagger)$$

The m families form a basis of $\mathbf{R}^n$. For all $l = 1, \cdots, m$ and $q = 0, \cdots, k_l - 1$, define

$$\tau^l_{q+1} = A^q b_l + \sum_{j=0}^{q} \sum_{i=1}^{m} \gamma^{il}_{k_l - j} A^{q-j} b_i. \quad (**)$$

Note that for all $l = 1, \cdots, m$,

$$\tau^l_{k_l+1} = A^{k_l} b_l + \sum_{j=0}^{k_l} \sum_{i=1}^{m} \gamma^{il}_{k_l - j} A^{k_l - j} b_i = 0.$$

Since, from $(\dagger)$ and $(\ddagger)$, only previously selected vectors are present in the second term of RHS of $(**)$, n vectors τ^l_{q+1} are linearly independent and form a basis. Define a nonsingular matrix:

$$T^{-1} = [\tau^1_{k_1}\ \tau^1_{k_1-1}\ \cdots\ \tau^1_1\ \tau^2_{k_2}\ \cdots\ \tau^m_{k_m}\ \cdots\ \tau^m_1].$$

The similarity transform defined by this matrix gives controllable canonical form:

$$\tilde{A} = TAT^{-1} \quad \tilde{B} = TB.$$

Fact: For all $l = 1, \cdots, m$ and $q = 0, \cdots, k_l - 1$, τ^l_{q+1} is the state at time $q+1$ produced by the recursion equation $x(j+1) = Ax(j) + Bu(j)$ due to $x(0) = 0$ and $u^l(j) = \begin{bmatrix} \gamma^{1l}_{k_l - j} \\ \vdots \\ \gamma^{ml}_{k_l - j} \end{bmatrix}$ with $\gamma^{ll}_{k_l}$ replaced by 1.

From this fact, for all $l = 1, \cdots, m$,

$$\tau^l_1 = Bu^l(0) = b_l + \sum_{i=1}^{l-1} \gamma^{il}_{k_l} b_i$$

and for all $l = 1, \cdots, m$ and $q = 0, \cdots, k_l - 1$,

$$A\tau^l_{q+1} = \tau^l_{q+2} - Bu^l(q+1) \quad with\ \tau^l_{k_l+1} = 0.$$

Moreover,

$$L^{-1} := [u^1(0)\ u^2(0)\ \cdots\ u^m(0)]$$

is an upper triangular nonsingular matrix with diagonal elements equal to 1 ($\gamma_{k_l}^{ll}$ must be replaced by 1 and $\gamma_{k_l}^{il} = 0$ for $i = l+1, \cdots, m$). Hence,

$$B = [\tau_1^1 \; \tau_1^2 \; \cdots \; \tau_1^m] L$$

and for all $l = 1, \cdots, m$ and $q = 0, \cdots, k_l - 1$,

$$A\tau_{q+1}^l = \tau_{q+2}^l - [\tau_1^1 \; \tau_1^2 \; \cdots \; \tau_1^m] L a^l(q+1) \quad \textit{with} \; \tau_{k_l+1}^l = 0$$

and L is an upper triangular matrix with diagonal elements equal to 1. Since L is nonsingular, $\mathcal{R}(B) = span\{\tau_1^1, \cdots, \tau_1^m\}$. Let $L = [\beta^{il}]_{i,l=1}^m$. Then $\beta^{ll} = 1$ and $\beta^{il} = 0$ for $i > l$, and for all $l = 1, \cdots, m$,

$$b_l = \sum_{i=1}^{l-1} \beta^{il} \tau_1^i + \tau_1^l.$$

Now the general form of $\tilde{B}$ follows.

Let $La^l(q+1) = \begin{bmatrix} \alpha_{k_l-q}^{1l} \\ \vdots \\ \alpha_{k_l-q}^{ml} \end{bmatrix}$. Then for all $l = 1, \cdots, m$, $q = 0, \cdots, k_l - 1$,

$$A\tau_{q+1}^l = \tau_{q+2}^l - \sum_{i=1}^{m} \alpha_{k_l-q}^{il} \tau_1^i.$$

From the matrix representation theorem, $\tilde{A}$ follows.

6.3 Observable Canonical Form

By duality, (C, A) is in observable canonical form iff (A^*, C^*) is in controllable canonical form.

6.4 Poles and Zeros

Lemma: Suppose $\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$ has full column normal rank. Then z_0 is an invariant zero iff $\exists\ x \neq 0$ and u such that

$$\begin{bmatrix} A-z_0I & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0.$$

Moreover, if $u = 0$, z_0 is also a unobservable mode.

Proof: $(\Rightarrow)$ Obvious

$(\Leftarrow)$ $\exists \begin{bmatrix} x \\ u \end{bmatrix} \neq 0$ such that

$$\begin{bmatrix} A-z_0I & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0.$$

Suppose $x = 0$. Then from full column normal rank assumption, $\begin{bmatrix} B \\ D \end{bmatrix} u = 0$ and thus $\begin{bmatrix} x \\ u \end{bmatrix} = 0$ (contradiction).

Finally note that if $u = 0$, then

$$\begin{bmatrix} A-z_0I \\ C \end{bmatrix} x = 0$$

and z_0 is a unobservable mode by PBH test.

Lemma: Suppose $\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$ has full row normal rank. Then z_0 is an invariant zero iff $\exists\ y \neq 0$ and v such that

$$[y^* \ v^*] \begin{bmatrix} A-z_0I & B \\ C & D \end{bmatrix} = 0.$$

Moreover, if $v = 0$, z_0 is also a uncontrollable mode.

Lemma: G has full column (row) normal rank iff $\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$ has full column (row) normal rank.

Proof: Notice that

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ C(A - sI)^{-1} & I \end{bmatrix} \begin{bmatrix} A - sI & B \\ 0 & G(s) \end{bmatrix}$$

where

$$G(s) = C(sI - A)^{-1}B + D.$$

Hence

$$normalrank \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + normalrank(G(s)).$$

Theorem: Let R be a minimal realisation of G. Then λ is an eigenvalue of A with multiplicity m iff λ is a pole of H of order m.

Proof: We will only prove the case that the transmission zero is not a pole of $G(s)$. Then z_0 is not an eigenvalue of A due to minimality. Note From the proof of the previous lemma and the fact that z_0 is not a pole,

$$normalrank \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} = n + normalrank(G(z_0)).$$

Hence,

$$rank \begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} < normalrank \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$$

iff $rank G(z_0) < normalrank G(s)$.

Corollary: Every transmission zero of G is an invariant zero of all its realisations and every pole of G is a pole of all its realisations.

Lemma: Let $[A, B, C, D]$ be a minimal realisation of G. If $u(t) = u_0 e^{\lambda t}$ where λ is not a pole of G and any u_0, then the output due to u and $x(0) = (\lambda I - A)^{-1} B u_0$ is

$$y(y) = G(\lambda) u_0 e^{\lambda t}.$$

Proof: In Laplace domain,

$$Y(s) = C(sI - A)^{-1}x(0) + C(sI - A)^{-1}BU(s) + DU(s)$$

$$= C(sI - A)^{-1}x(0) + C(sI - A)^{-1}Bu_0(s - \lambda)^{-1} + Du_0(s - \lambda)^{-1}$$

$$= C(sI - A)^{-1}(x(0) - (\lambda I - A)^{-1}Bu_0) + G(\lambda)u_0(s - \lambda)^{-1} = G(\lambda)u_0(s - \lambda)^{-1}.$$

Corollary: Let $[A, B, C, D]$ be a minimal realisation of G. Suppose z_0 is a transmission zero but not a pole. Then for $u_0 \neq 0$, the output due to $x(0) = (z_0 I - A)^{-1} B u_0$ and $u = u_0 e^{z_0 t}$ is zero.

This is called the transmission blocking property.

If z_0 is blocking zero, the transmission blocking occurs for $u(s) = \frac{u_0}{s - z_0}$ where u_0 is any r-D vector.

Lemma: Suppose $G = [A, B, C, D]$ is a square transfer function matrix with D nonsingular and suppose z_0 is an eigenvalue of A (not necessarily minimal). Then $\exists\ x_0$ such that

$$(A - BD^{-1}C)x_0 = z_0 x_0, \quad Cx_0 \neq 0$$

iff $\exists\ u_0 \neq 0$ such that

$$G(z_0)u_0 = 0.$$

Proof: $(\Leftarrow)$ $G(z_0)u_0 = 0$ implies that

$$G^{-1}(s) = \left[\begin{array}{cc} A - BD^{-1}C & -BD^{-1} \\ D^{-1}C & D^{-1} \end{array} \right]$$

has a pole at z_0 which is not observable, Then $\exists\ x_0$ such that

$$(A - BD^{-1}C)x_0 = z_0 x_0$$

and

$$Cx + 0 \neq 0.$$

$(\Rightarrow)$ Set $u_0 = -D^{-1}Cx_0 \neq 0$. Then

$$(z_0 I - A)x_0 = -BD^{-1}Cx_0 = Bu_0.$$

Using this equality, one gets

$$G(z_0)u_0 = C(z_0 I - A)^{-1}Bu_0 + Du_0 = Cx_0 - Cx_0 = 0.$$

This lemma implies z_0 is a zero of an invertible $G(s)$ iff it is a pole of $G^{-1}(s)$.

Chapter 7

Linear Static State Feedback and Linear Estimation

7.1 Linear Static State Feedback

Consider the plant:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

and linear static state feedback control:

$$u(t) = Fx(t) + v(t).$$

Then the closed loop system is

$$\dot{x}(t) = A_f x(t) + Bv(t)$$
$$y(t) = Cx(t)$$

where

$$A_f = A + BF.$$

Question (pole placement problem): given a monic polynomial

$$\pi(s) = s^n + \pi_n s^{n-1} + \cdots + \pi_1$$

does there exist F such that

$$\chi_{A_f} = \pi?$$

Theorem: (A, B) controllable iff $\exists F$ such that $\chi_{A_f} = \pi$.

Proof: $(\Rightarrow)$ (A, B) controllable

$\Rightarrow$ $\exists$ F_1 such that $(A + BF_1, b_1)$ controllable.

$\Rightarrow$ $\exists$ f_2 such that

$$\chi_{A+BF_1+b_1 f_2} = \pi.$$

Let $F_2 = [f_2 \; 0 \; \cdots \; 0]$ such that $BF_2 = b_1 f_2$. Setting $F = F_1 + F_2$ and $A_f = A + BF = A + B_1 F_1 + b_1 f_2$, we get $\chi_{A_f} = \pi$.

$(\Leftarrow)$ Suppose (A, B) is not controllable. (A, B) has an uncontrollable hidden mode; i.e. $\exists$ $\lambda \in A$ and $\eta \in \mathbf{C}^n$ such that

$$\eta^* A = \eta^* \lambda \quad \text{and} \quad \eta^* B = 0.$$

Hence, $\exists$ $\eta \in \mathbf{C}^n$ such that for all F

$$\eta^*(A + BF) = \eta^* \lambda.$$

$\Rightarrow$ $A_f = A + BF$ has the fixed eigenvalue λ.

7.2 Linear Estimation

WLOG, consider

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t).$$

Linear State Estimator (Linear Observer)

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + L(\tilde{y}(t) - y(t))$$

where the predicted output $\tilde{y}(t) = C\tilde{x}(t)$.

$$\Downarrow$$

$$\dot{\tilde{x}}(t) = (A + LC)\tilde{x}(t) + Bu(t) - Ly(t).$$

Notice that the observer gives the state estimate $\tilde{x}$ from the I/O pair (u, y).

Define state estimation error

$$e(t) = \tilde{x}(t) - x(t)$$

$$\Downarrow$$

$$\dot{e}(t) = (A + LC)e(t).$$

Question: given a monic polynomial

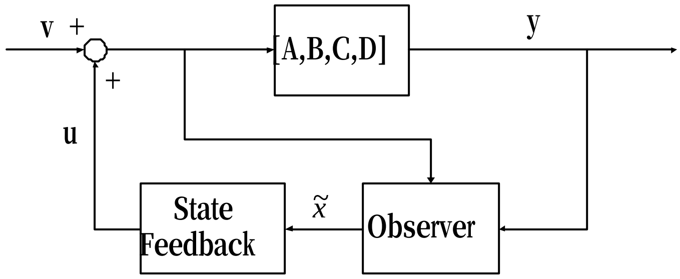
$$\pi(s) = s^n + \pi_n s^{n-1} + \cdots + \pi_1$$

does there exist L such that

$$\chi_{A+LC} = \pi?$$

By duality, we have the following theorem:
Theorem: (C, A) observerble iff $\exists L$ such that $\chi_{A+LC} = \pi$.

7.3 State Feedback of Estimated State



The state representation of the closed loop:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A + LC \end{bmatrix}\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v(t)$$

$$y(t) = [C\ 0]\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}.$$

Hence

$$G(s) = C(sI - A - BF)^{-1}B.$$

Separation Principle: the family of poles of the closed loop system is the union of those of state feedback system and state estimator.

Chapter 8

Advanced Topics

8.1 Matrix Fraction Description

8.2 Polynomial Matrix Description

8.3 Factorization

8.4 Linear Time-Varying Systems