LECTURE NOTE III

Chapter 4
Linear Programming

Ref: Luenberger, D. G., "Linear and Nonlinear Programming,: 2nd Ed., Addison-Wesley, 1984

- Objective function and constraints are *linear*.
- *Feasible region*: the region that satisfies all constrains
- *Unique optimal solution* vs. *alternative* (or *multiple) optimal solution*
- *Unbounded optimum*: infinite objective function value ($\pm\infty$) $\rightarrow$ due to insufficient constraints
- An optimal solution to LP, if it exists, will be on one of the *vertices* generated by constraints

Example: Let x_1 and x_2 denote the number of grade-1 and grade-2 inspectors assigned to inspection. The grade-1 inspector can inspect 200 pieces with error rate of 2% and 120 pieces with error rate of 5% for grade-2 inspector, daily. The hourly wages are \$4 and \$3 for grade-1 and grade-2, respectively. The erroneous inspection costs \$2 for each piece. Since the number of available inspectors in each grade is limited, we have following constraints:

$x_1 \le 8$ (grade-1) and $x_2 \le 10$ (grade-2)

The company requires at least 1800 pieces be inspected daily. Formulate the problem to minimize the cost for the company.

<*Solution*>

Total cost for company (objective function):

$$Z = (\$4/hr \times 8hr + \$2/ea \times 200ea \times 0.02)x_1$$
$$+ (\$3/hr \times 8hr + \$2/ea \times 120ea \times 0.05)x_2$$
$$= 40x_1 + 36x_2$$

Constrains: $x_1 \le 8$, $x_2 \le 10$, $200x_1 + 120x_2 \ge 1800$

$x_1 \ge 0$, $x_2 \ge 0$

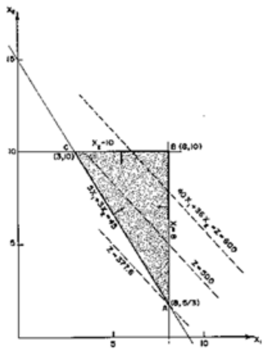


Figure 4.3. Graphical solution of Example 4.3.

Graphical interpretation:

feasible region: shaded area (satisfying all constraints)

optimum point: at point A, (8, 1.6), the objective function Z has the minimum of 377.6.

optimum solution: x_1=8, x_2=1.6 (one of grade-2 inspector is utilized only 60%)

(If not possible choose x_2=2.)

I. Basic Properties

1. Standard form:

$$\min_x (c_1x_1 + c_2x_2 + \cdots + c_nx_n)$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

and $x_1 \ge 0,\ x_2 \ge 0, \cdots,\ x_n \ge 0$.

Or in vector-matrix form,

$$\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$$

subject to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq 0$

(Without loss of generality, let $\boldsymbol{b} \geq \mathbf{0}$)

where the *coefficient matrix* **A** is *m*x*n*,

the *decision vector* **x** is *n*x1,

the *cost (profit) vector* **c** is *n*x1, and

the *requirement vector* **b** is *m*x1.

(*n* optimization variables and *m* constraints with scalar objective function)

▶ **Conversion to the standard form for the different types of constraints**

<Ex. 1>

$$\begin{cases} \mathbf{Ax} \leq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases} \Rightarrow \begin{cases} \mathbf{Ax} + \mathbf{s} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{s} \geq \mathbf{0} \end{cases}$$

(**s** : "*slack variable*")

<Ex. 2>

$$\begin{cases} \mathbf{Ax} \geq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases} \Rightarrow \begin{cases} \mathbf{Ax} - \mathbf{s} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{s} \geq \mathbf{0} \end{cases}$$

(*s* : "*surplus variable*")

cf) If slack or surplus variable is 0, the inequality constraint is *active*, and if they are positive, then the constraint is *binding*.

<Ex. 3>

$$-\infty < \mathbf{x} < \infty \Rightarrow \begin{cases} \mathbf{x} = \mathbf{u} - \mathbf{v} \\ \mathbf{u} \geq \mathbf{0} \text{ and } \mathbf{v} \geq \mathbf{0} \end{cases}$$

(replace **x** with **u** - **v**)

<Ex. 4> Maximization of *Z* is equivalent to minimization of (*-Z*).

<Ex. 5>

$$\begin{aligned} &\max_x (x_1 - 2x_2 + 3x_3) \\ &\text{subject to} \\ &x_1 + x_2 + x_3 \leq 7 \\ &x_1 - x_2 + x_3 \geq 2 \\ &3x_1 - x_2 - 2x_3 = -5 \\ &\text{and } x_1 \geq 0,\ x_2 \geq 0. \end{aligned} \longrightarrow \begin{aligned} &\min_x (-x_1 + 2x_2 - 3x_4 + 3x_5) \\ &\text{subject to} \\ &x_1 + x_2 + x_4 - x_5 + x_6 = 7 \\ &x_1 - x_2 + x_4 - x_5 - x_7 = 2 \\ &-3x_1 + x_2 + 2x_4 - 2x_5 = 5 \\ &\text{and } x_i \geq 0 \ (i = 1,2,4,5,6,7) \end{aligned}$$

2. Basic Feasible Solution

- $\mathbf{Ax} = \mathbf{b}$ (constraints)

Let $\mathbf{A} \in \mathrm{R}^{m \times n}$, $\mathbf{x} \in \mathrm{R}^n$, and $\mathbf{b} \in \mathrm{R}^m$.

⇒ *n* variables with *m* equations: (*n-m*) degrees of freedom (*n*>*m*)

⇒ *Full rank assumption*: *m* rows of **A** are linearly independent.

cf) if *n*=*m*, unique solution exists if *rank*(**A**)=*m*.

if *n*<*m*, it is a over-determined system and there is no solution if *rank*(**A**)>*n*.

if *n*>*m*, it is a under-determined system and there are many solutions.

(select the optimal among many solutions)

- Let $\mathbf{A} = [\mathbf{B} \quad \mathbf{N}]$ and $\mathbf{x} = [\mathbf{x}_1 \quad \mathbf{x}_2]^T$
  where $\mathbf{B}$ is mxm, $\mathbf{N}$ is mx$(n\text{-}m)$, $\mathbf{x}_1$ is mx1, and $\mathbf{x}_2$ is $(n\text{-}m)$x1.

$$\mathbf{Ax} = \mathbf{Bx}_1 + \mathbf{Nx}_2 = \mathbf{b}$$

- Only m variables can be fixed and others can have *any values* to satisfy the m-constraints.
  By letting $\mathbf{x}_2 = 0$ and rearrange the variable indices so that the first m variables is $\mathbf{x}_B$,

$$\mathbf{x}_1 = \mathbf{B}^{-1}\mathbf{b} \equiv \mathbf{x}_B \qquad \text{(\textit{Basic variable})}$$

$$\mathbf{x}_2 = 0 \equiv \mathbf{x}_N \qquad \text{(\textit{Nonbasic variable})}$$

$$\mathbf{x} = \left[\mathbf{x}_B \quad \mathbf{0}\right]^T \qquad \text{(\textit{Basic solution})}$$

If $\mathbf{x}_B \geq \mathbf{0}$, $\mathbf{x} = [\mathbf{x}_B\ \mathbf{0}\ ]^T$ is the *basic feasible solution*.

If some of the elements of $\mathbf{x}_B$ are zero, $\mathbf{x} = [\mathbf{x}_B\ \mathbf{0}\ ]^T$ is the *degenerate basic feasible solution*.

3. The Fundamental Theorem of LP

$$\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x}$$

$$\text{subject to} \quad \mathbf{Ax} = \mathbf{b} \quad \text{and } \mathbf{x} \geq 0$$

Assume $rank\left(\mathbf{A} \in \mathrm{R}^{m\times n}\right) = m$,

i) if there is a feasible solution, there is a basic feasible solution;

ii) if there is an optimal feasible solution, there is an optimal basic feasible solution.

⇒ It is sufficient to search only over the basic feasible solutions!

$$\text{No. of basic feasible solutions} \leq \binom{n}{m} = \frac{n!}{(n-m)!} \quad \text{(some are not feasible)}$$

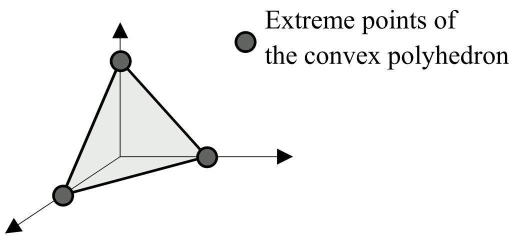
4. Geometric Meaning of Basic Feasible Solutions

<Ex. 1>

$x_1 + x_2 + x_3 = 1 \quad (m{=}1)$

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$

$$\mathbf{x}_B = \begin{bmatrix}1\\0\\0\end{bmatrix}, \text{ or } \begin{bmatrix}0\\1\\0\end{bmatrix}, \text{ or } \begin{bmatrix}0\\0\\1\end{bmatrix}$$



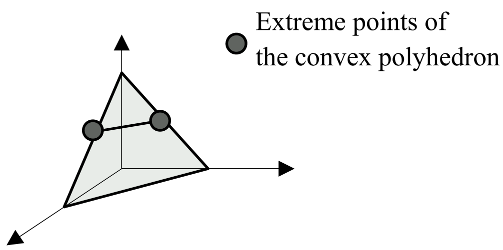
<Ex. 2>

$x_1 + x_2 + x_3 = 1$

$2x_1 + 3x_2 = 1 \quad (m{=}2)$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

$$\mathbf{x}_B = \begin{bmatrix}1/2\\0\\1/2\end{bmatrix}, \text{ or } \begin{bmatrix}0\\1/3\\2/3\end{bmatrix}$$



⇒ Solution can be found among basic feasible solutions (vertices).

$\{\boldsymbol{x} \mid \boldsymbol{Ax} = \boldsymbol{b}\}$ ⇒ *Linear variety*, *Affine space* (Not a linear space → have not origin)

$\{\boldsymbol{x} \mid \boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ ⇒ *Convex polytope* (has extreme points)

Extreme point : A point $\boldsymbol{x}$ in a convex set C is said to be an *extreme point* of C if there are no two distinct point $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in C such that $x = \alpha x_1 + (1-\alpha)x_2$ for some $0 < \alpha < 1$.

Theorem (Equivalence of extreme points and basic solutions): Let $\mathbf{A}$ be an mxn matrix of rank m and $\mathbf{b}$ an m-vector. Let $\mathbf{K}$ be the convex polytope consisting of all n-vector $\mathbf{x}$ satisfying $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. A vector $\mathbf{x}$ is an *extreme point* of $\mathbf{K}$ if and only if $\mathbf{x}$ is a *basic feasible solution* of $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

Corollary 1: If the convex set K corresponding to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ is nonempty, it has at least one extreme point.

Corollary 2: If there is a finite optimal solution to a linear programming problem, there is a finite solution which is an extreme point of the constraint set.

Corollary 3: The constraint set K corresponding to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ possesses at most a finite number of extreme points.

Corollary 4: If the convex polytope K corresponding to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$ is bounded, then K is a convex polyhedron, that is, K consists of points that are convex combinations of a finite number of points.

5. Searching the Basic Solution, Pivot Operation

Elementary row operation : Add the constant multiple of a row to another row

(Does not alter the solution of the linear system equation)

$$\text{Let } \mathbf{E}_1 = \begin{bmatrix} 1 & 0 & \cdots & d_{1i} & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \quad \text{(mxm matrix)}$$

$\mathbf{E}_1\mathbf{Ax} = \mathbf{E}_1\mathbf{b}$ $\Rightarrow$ Addition of d_{1i} times of the i-th row to the first row

Pivot operation : Reduce the coefficient of a specific variable to unity in on of the equation and zero elsewhere. Let $\mathbf{E} = \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \cdots \mathbf{E}_m$ which will perform a pivot operation and consists of m-elementary row operations where

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & \cdots & -a_{1q}/a_{pq} & \cdots & 0 \\ 0 & 1 & \cdots & -a_{2q}/a_{pq} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/a_{pq} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -a_{mq}/a_{pq} & \cdots & 1 \end{bmatrix} \quad \text{(mxm matrix)}$$

Then $\mathbf{EAx} = \mathbf{Eb}$ will have a form that is pivoted by the pq-element.

By m-pivot operations (or by *Gauss-Jordan elimination*) for $\mathbf{Ax} = \mathbf{b}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1,m+1} & \cdots & y_{1n} \\ 0 & 1 & \cdots & 0 & y_{2,m+1} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & y_{m,m+1} & \cdots & y_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{bmatrix}$$

$\Rightarrow$ *Row Echelon Form*, or *Canonical Form*

⇒ Set $x_{m+1} = x_{m+2} = \cdots = x_n = 0$. Then $x_1 = y_{10}$, $x_2 = y_{20}$, $\cdots$ $x_m = y_{m0}$,

$$\begin{bmatrix} \mathbf{x}_B & \mathbf{0} \end{bmatrix}^T = \begin{bmatrix} y_{10} & \cdots & y_{m0} & \mathbf{0} \end{bmatrix}^T \quad \text{(A basic Solution)}$$

Alternatively,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow \begin{bmatrix} y'_{1,m+1} & \cdots & y'_{1n} & 1 & 0 & \cdots & 0 \\ y'_{2,m+1} & \cdots & y'_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y'_{m,m+1} & \cdots & y'_{mn} & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y'_{10} \\ y'_{20} \\ \vdots \\ y'_{m0} \end{bmatrix}$$

⇒ Set $x_1 = x_2 = \cdots = x_{n-m} = 0$. Then $x_{m+1} = y'_{10}$, $x_{m+2} = y'_{20}$, $\cdots$ $x_n = y'_{m0}$,

$$\begin{bmatrix} \mathbf{0} & \mathbf{x}_B \end{bmatrix}^T = \begin{bmatrix} \mathbf{0} & y'_{10} & \cdots & y'_{m0} \end{bmatrix}^T \quad \text{(An \textit{alternative basic solution})}$$

⇒ The ERO's are applied to both matrix **A** and vector **b**. Therefore, apply ERO's for the following augmented matrix.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

6. Finding a Minimum Feasible Solution While Preserving Feasibility

Adjacent basic feasible solution: It is a basic feasible solution which differs from the present basic feasible solution in exactly one basic variable. (Adjacent vertex or extreme point which has different objective function value ⇒ could be better or worse)

⇒ Select the better basic feasible solution (among mx(n-m) solutions at most)
(If no adjacent basic feasible solution is better than the present, optimal!)

⇒ Exchange one variable in the basic variable set and on in nonbasic variable set in a way that the objective function value is improved.

⇒ Once the variable to exchange is selected (p-th variable in the basic and q-th in the nonbasic), pivot by pq-th element!

▶ Row echelon form

$\mathbf{I}\,\mathbf{x}_B + \mathbf{B}^{-1}\mathbf{N}\,\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b}$ (where $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\,\mathbf{x}_N$ and $\mathbf{x}_N = \mathbf{0}$)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1,m+1} & \cdots & y_{1n} \\ 0 & 1 & \cdots & 0 & y_{2,m+1} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & y_{m,m+1} & \cdots & y_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{m0} \end{bmatrix}$$

($\mathbf{B}^{-1}\mathbf{N}$: the block $y_{1,m+1} \ldots y_{mn}$; $\mathbf{x}_B$: upper part of $\mathbf{x}$; $\mathbf{x}_N$: lower part of $\mathbf{x}$; $\mathbf{B}^{-1}\mathbf{b}$: $y_{10} \ldots y_{m0}$)

▶ Objective function

$J = \mathbf{c}_B^T\,\mathbf{x}_B + \mathbf{c}_N^T\,\mathbf{x}_N = \mathbf{c}_B^T\,\mathbf{x}_B$

▶ Assume x_q from nonbasic variables is changed from 0 to 1: Impact of x_q on objective

$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\,\mathbf{x}_N = \mathbf{B}^{-1}\mathbf{b} - \mathbf{d}_q x_q = \mathbf{B}^{-1}\mathbf{b} - \mathbf{d}_q$, $\qquad (x_q = 1)$

where $\mathbf{d}_q$ is q-th column of $\mathbf{B}^{-1}\mathbf{N}$.

$J' = \mathbf{c}_B^T\,(\mathbf{B}^{-1}\mathbf{b} - \mathbf{d}_q) + \mathbf{c}_N^T\,\mathbf{x}_N = \mathbf{c}_B^T\,(\mathbf{B}^{-1}\mathbf{b} - \mathbf{d}_q) + c_q$

▶ Improvement on objective by including x_q in the basic (*Inner product rule*)

$\Delta J = J' - J = [\mathbf{c}_B^T (\mathbf{B}^{-1}\mathbf{b} - \mathbf{d}_q) + c_q] - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b} = c_q - \mathbf{c}_B^T \mathbf{d}_q$

$$\bar{c}_q \triangleq c_q - \mathbf{c}_B^T \mathbf{d}_q$$

⇒ *relative cost (or profit)* of the nonbasic variable x_q

⇒ should be *negative* for improvement in minimization problem

⇒ should be *positive* for improvement in maximization problem

⇒ For minimization, consider a new basic variable which has the most improvement on objective function.

▶ Condition for optimality (Minimization)

$$\bar{c}_q \geq 0 \quad \forall q \quad (q = m+1, \cdots, n)$$

(A local minimum is the global minimum since it is LP.)

⇒ **Determining variable to enter basis (min.):**

Choose q so that $\bar{c}_q = \min_j \bar{c}_j < 0$

▶ Selection of the leaving element from the basic variable set

⇒ To achieve the greatest improvement on objective, the x_q should increase the objective function as much as possible, but $\mathbf{x}_B$ cannot be nonpositive in more than one element.

⇒ Increase x_q until any one of elements in $\mathbf{x}_B$ becomes zero ($x_p = 0$).

$$x_j = y_{j0} - \left(\mathbf{B}^{-1}\mathbf{N}\right)_{jq} x_q = y_{j0} - y_{jq}x_q \geq 0 \quad (j = 1, \cdots m)$$

− If y_{jq} is not positive, x_j becomes *more positive* as x_q increases. (cannot be nonbasic)

− Choose p so that the maximum increase in x_q without making more than one basic variable nonpositive.

⇒ **Determining variable to leave basis (min.):**

Choose p so that (y_{j0} / y_{jq}) is smallest among $y_{jq} > 0$ for $j = 1, \cdots, m$.

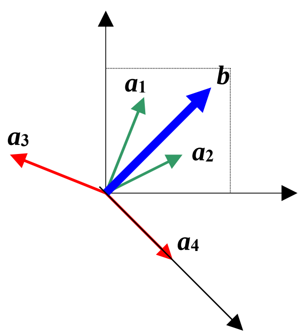
(*Minimum ratio rule*)

▶ Geometrical Interpretation of the basic feasible solution

$$\boldsymbol{Ax} = \boldsymbol{b} \quad \Rightarrow \quad \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Choose $\{\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_m\}$ as the basis so that

$$\mathbf{x}_B = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \end{bmatrix}^{-1} \mathbf{b} \quad (x_{Bi} > 0)$$



Assume m=2 and n=4.

A feasible solution defines a representation of $\boldsymbol{b}$ as a *positive combination* of $\boldsymbol{a}_i$'s. In this example, $\{\boldsymbol{a}_1, \boldsymbol{a}_3\}$, $\{\boldsymbol{a}_2, \boldsymbol{a}_4\}$ and $\{\boldsymbol{a}_3, \boldsymbol{a}_4\}$ cannot result in basic feasible solution.

II. SIMPLEX METHOD

Standard LP problem can be written as:

$$\min_x Z = \mathbf{c}_B^T\mathbf{x}_B + \mathbf{c}_N^T\mathbf{x}_N$$

Subject to $\mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b}$, $\mathbf{x}_B \geq \mathbf{0}$ and $\mathbf{x}_N \geq \mathbf{0}$

If **B** is the basis for a basic feasible solution ($\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} > \mathbf{0}$, $\mathbf{x}_N = \mathbf{0}$),

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\,\mathbf{x}_N$$

$$Z = \mathbf{c}_B^T\,\mathbf{x}_B + \mathbf{c}_N^T\,\mathbf{x}_N = \mathbf{c}_B^T(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{N}\,\mathbf{x}_N) + \mathbf{c}_N^T\,\mathbf{x}_N$$

$$= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N})\,\mathbf{x}_N = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + \mathbf{r}^T\,\mathbf{x}_N$$

1. Matrix Form of simplex method (*Tableau form*)

$$\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{c}^T & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{N} & \mathbf{b} \\ \mathbf{c}_B^T & \mathbf{c}_N^T & \mathbf{0} \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{I} & \mathbf{B}^{-1}\mathbf{N} & \mathbf{B}^{-1}\mathbf{b} \\ \mathbf{0} & \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N} & -\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} \end{bmatrix} \quad \text{(by ERO)}$$

Objective function value (→ $-\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$)

relative (reduced) cost coeff. (→ $\mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}$)

<Procedure>

1. Select *m*-initial basic variables.
2. Convert the tableau into the row echelon form by ERO's.
3. Choose a column which has the largest negative value of the reduced cost coefficient (Select *q*).
4. Choose a pivot element in the *q*-th column by the minimum ratio rule.
5. Include x_q as a basic variable and x_p becomes a nonbasic variable.
6. Repeat the step 2-6 until all the reduced costs are positive (Minimization problem)

▶ Maximization problem

1. Replace *Z* with –*Z*.
2. Or, choose largest positive value of the *relative profit coefficient* in step 3 and repeat the procedure until all the reduced costs are negative.

▶ *Alternative Optima*: If the reduced costs for nonbasic variables are not positive at the optimal solution, this indicates the alternative solutions which will not change the optimum value.

▶ *Unbounded Optimum*: If the minimum ratio rule cannot be determined (all negative), this indicate the unbounded optimum.

▶ *Degeneracy*: If a basic feasible solution contains zeros for one or more basic variables, this indicates the *degenerate basic feasible solution*. It occurs if two or more rows tie for the minimum ratio, or one or more elements of the right-hand side in the constraints (b_i's) in original LP problem are zero.

→ Perturb the pivot entry in the requirement vector (**b**) by a small amount (ε), then proceed as normal.

▶ *Cycling*: If the degeneracy occurs (no improvement with new basic variable), no improvement in the objective function is achieved for a while. (Reducing calculation efficiency) If the iteration goes on indefinitely without improving the objective function value, it is called *classical cycling* or *cycling*.

→ It is not likely in practice, but *computer cycling* may occur!

Example:

$$\max Z = 3x_1 + x_2 + 3x_3$$

$$\text{subject to} \begin{cases} 2x_1 + x_2 + x_3 \le 2 \\ x_1 + 2x_2 + 3x_3 \le 5 \\ 2x_1 + 2x_2 + x_3 \le 6 \end{cases} \quad x_i \ge 0 \quad (i = 1, 2, 3)$$

Introducing three nonnegative slack variables, the initial tableau

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{b}$	$\mathbf{a}_{i0}/\mathbf{a}_{ij}$
	2	1	1	1	0	0	2	2/2
	1	2	3	0	1	0	5	5/1
	2	2	1	0	0	1	6	6/2
$\mathbf{r}^T$	-3	-1	-3	0	0	0	0	

Select pivot element as 1st column and 1st row and perform ERO's.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{b}$	$\mathbf{a}_{i0}/\mathbf{a}_{ij}$
	1	1/2	1/2	1/2	0	0	1	2
	0	3/2	5/2	-1/2	1	0	4	8/5
	0	1	0	-1	0	1	4	big
$\mathbf{r}^T$	0	1/2	-3/2	3/2	0	0	3	

Select pivot element as 3rd column and 2nd row and perform ERO's.

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	$\mathbf{b}$	$\mathbf{a}_{i0}/\mathbf{a}_{ij}$
	1	1/5	0	2/5	-1/5	0	1/5	
	0	3/5	1	-1/5	1/5	0	8/5	
	0	1	0	-1	0	1	4	
$\mathbf{r}^T$	0	11/10	0	6/5	3/10	0	27/5	

All the relative costs are positive: Optimal solution is obtained.

x_1=1/5, x_3=8/5, x_2=0, x_4=0, x_5=0, x_6=4, and Z= –27/5.

III. TWO-PHASE SIMPLEX METHOD

- To start the *simplex method*, an *initial feasible solution in canonical form is needed*.
- Generally, the constraints are not given in the canonical form.
  → Need to solve the linear equation to find a basic feasible solution

1. *Artificial Variable*

If basic variables are not readily available from each constraint, add new *artificial variable* for each constraints to form a basic solution.

<Example>

$\min Z = -3x_1 + x_2 + x_3$

$$\text{s. t.}\quad \begin{aligned} x_1 - 2x_2 + x_3 &\le 11 \\ -4x_1 + 2x_2 + 2x_3 &\ge 3 \\ 2x_1 - x_3 &= -1, \qquad x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \end{aligned}$$

⇒ To standard form (with *slack/surplus variable*)

$$\begin{aligned} x_1 - 2x_2 + x_3 + x_4 &= 11 \\ -4x_1 + 2x_2 + 2x_3 - x_5 &= 3 \\ -2x_1 + x_3 &= 1, \qquad x_1 \sim x_5 \ge 0 \end{aligned}$$

→ x_4 can be a basic variable. Need two more basic variables!

⇒ Add 2 artificial variables (x_6, x_7)

$$
\begin{aligned}
x_1 - 2x_2 + x_3 + x_4 \qquad\qquad &= 11 \\
-4x_1 + 2x_2 + 2x_3 \qquad - x_5 + x_6 \qquad &= 3 \\
-2x_1 \qquad + x_3 \qquad\qquad + x_7 &= 1, \quad x_1 \sim x_7 \geq 0
\end{aligned}
$$

⇒ Basic feasible solution : $x_4 = 11$, $x_6 = 3$, $x_7 = 1$, $x_1 \sim x_3$, $x_5 = 0$

(It is not feasible to original problem since x_6 and x_7 are not zero.)

⇒ Need to make x_6 and x_7 zero ASAP for the feasible solution to original problem!!

2. *Two-Phase Simplex Method*

<**Phase I**> To remove all the artificial variables, set an *artificial objective function*.

$$\min\ w = x_6 + x_7$$

$$
\begin{aligned}
\text{s. t.}\quad x_1 - 2x_2 + x_3 + x_4 \qquad\qquad &= 11 \\
-4x_1 + 2x_2 + 2x_3 \qquad - x_5 + x_6 \qquad &= 3 \\
-2x_1 \qquad + x_3 \qquad\qquad + x_7 &= 1, \quad x_1 \sim x_7 \geq 0
\end{aligned}
$$

→ If w becomes zero by simplex method ($x_6 = 0$ and $x_7 = 0$), then the solution is a basic feasible solution of the original problem. If not, the original problem is *infeasible*.

→ Start the initial tableau calculating reduced costs by making the reduced costs for artificial variables zero.

<**Phase II**> Find the optimum of the original problem with the solution from the Phase I without any artificial variables.

3. Variables With Upper Bounds

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x}$$

$$\text{Subject to } \mathbf{Ax} = \mathbf{b} \text{ and } \mathbf{0} \leq \mathbf{x} \leq \mathbf{h}$$

<**Method 1**> Change the problem to standard form

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x}$$

$$\text{Subject to } \mathbf{Ax} = \mathbf{b},\ \mathbf{x} + \mathbf{y} = \mathbf{h} \text{ and } \mathbf{x} \geq \mathbf{0},\ \mathbf{y} \geq \mathbf{0}$$

- The size of coefficient matrix is changed from mxn to $(m+n)$x$2n$.
- This transformation requires more memory and computation time.

<**Method 2**>

Definition: An *extended basic feasible solution* corresponding to the problem for the variables with upper bounds is a feasible solution for which $(n-m)$ variables are equal to either their lower (zero) or their upper bound; and the remaining m basic variables correspond to linearly independent column of **A**.

- If there is no extended basic feasible solution with m basic variables, then the constraints are too tight and there is no solution.
- Assume that every extended basic feasible solution is nondegenerate.
- *A variable at its lower bound* can only be increased, and an increase will be beneficial if the corresponding relative cost coefficient is *negative*.
- *A variable at its upper bound* can only be decreased, and the decrease will be beneficial if the corresponding relative cost coefficient is *positive*.
- ***Theorem*** (*Upper bound optimality*): An extended basic feasible solution is optimal for the upper bound problem if for the nonbasic variables x_j

$$r_j \geq 0 \text{ if } x_j = 0$$

$$r_j \leq 0 \text{ if } x_j = h_j$$

- Define new variables, $x_i^+ = x_i$ (previously x_i was at lower bound) and $x_i^- = h_i - x_i$ (previously x_i was at upper bound).
- Extended tableau

$$\begin{array}{cccccccc}
1 & 0 & \cdots & 0 & y_{1,m+1} & \cdots & y_{1n} & y_{10} \\
0 & 1 & \cdots & 0 & y_{2,m+1} & \cdots & y_{2n} & y_{20} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & y_{m,m+1} & \cdots & y_{mn} & y_{m0} \\
0 & 0 & \cdots & 0 & r_{m+1} & \cdots & r_n & -Z_0 \\
e_1 & e_2 & \cdots & e_m & e_{m+1} & \cdots & e_n &
\end{array}$$

- The e_i is either + or – depending on the current solution, x_i^+ or x_i^-.
- **Strategy**:
  1. Determine a nonbasic variable $x_j^{e_j}$ for which r_j<0. If no such variable exists, stop; the current solution is optimal. (j-th column is selected for basic)
  2. Based on the three numbers, a) h_j, b) $\min\limits_{i,y_{ij}>0} y_{i0}/y_{ij}$, c) $\min\limits_{i,y_{ij}<0} (y_{i0}-h_i)/y_{ij}$, follow the update strategy according to which number is smallest.
     Case a) The variable x_j goes to its opposite bound: Subtract h_j times column j from last column and change signs of column j. The basis does not change and no pivot is required.
     Case b) The i-th basic variable returns to its old bound: Pivot on the ij-th element.
     Case c) The i-th basic variable goes to its opposite bound: Subtract h_i from y_{i0} and change signs of y_{ii} and e_i. Pivot on the ij-th element.
  3. Return to step 1.

- **Example**:

$$\min Z = 2x_1 + x_2 + 3x_3 - 2x_4 + 10x_5$$

$$\text{Subjec to } \; x_1 + x_3 - x_4 + 2x_5 = 5$$

$$x_2 + 2x_3 + 2x_4 + x_5 = 9$$

$$0 \le x_1 \le 7,\; 0 \le x_2 \le 10,\; 0 \le x_3 \le 1,\; 0 \le x_4 \le 5,\; 0 \le x_5 \le 3$$

Original tableau is

	a_1	a_2	a_3	a_4	a_5	b	a)	b)	c)
	1	0	1	(-1)	2	5	h_j=5	-	(5-7)/(-1)=2
	0	1	2	2	1	9		9/2	-
$\mathbf{r}^T$	0	0	-1	-2	5	-19			
e_i	+	+	+	+	+		j=4	i=2	i=1

r_3=3–1x2-1x1= –1, r_4= –2+1x2–2x1= –2, r_5=10–2x2–1x1= 5

The c) is the minimum. Case c) should be applied. Before pivoting,

	a_1	a_2	a_3	a_4	a_5	b	a)	b)	c)
	-1	0	1	(-1)	2	-2			
	0	1	2	2	1	9			
$\mathbf{r}^T$	0	0	-1	-2	5	-19			
e_i	-	+	+	+	+				

b_i= 7–5= –2

After pivoting,

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	**b**	**a)**	**b)**	**c)**
	1	0	-1	1	-2	2	h_j=1	-	(2-
	-2	1	4	0	5	5		5/4	-
$\mathbf{r}^T$	2	0	-3	0	1	-15			
e_i	-	+	+	+	+		j=3	i=2	i=4

The a) has the minimum. Case a) should be applied. (**b**=**b**– $h_j\mathbf{a}_3$, $\mathbf{a}_3$=–$\mathbf{a}_3$, no pivot.)

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	**b**	**a)**	**b)**	**c)**
	1	0	1	1	-2	3			
	-2	1	-4	0	5	1			
$\mathbf{r}^T$	2	0	3	0	1	-			
e_i	-	+	-	+	+		j=3		

There is no r_i<0 for nonbasic variables. The optimum is obtained!

$$x_2 = y_{20} = 1,\ \ x_4 = y_{40} = 3,\ \ x_1 = h_1 = 7\ (\because e_1 = -)$$

$$x_3 = h_3 = 1\ (\because e_3 = -),\ \text{and}\ \ x_5 = 0\ (\because e_5 = +)$$

IV. REVISED SIMPLEX METHOD

- For large size problem (>5000 constraints), there are problems in memory and computation time,
- For efficient computer implementation, some modifications of the simplex method are needed.
- Other columns besides pivot column are not explicitly used. (if n>>m, waste of computation)
- The pivoting will be applied to $\mathbf{B}^{-1}$ and $\mathbf{y}_q$, not the whole tableau.

<Procedure> (Minimization case)

Given a current basis $\mathbf{B}^{-1}$ and the current solution $\mathbf{x}_B = \mathbf{y}_0 = \mathbf{B}^{-1}\mathbf{b}$,

1. Calculate current *reduced cost coefficients*: $\bar{\mathbf{c}}_N^T = \mathbf{c}_N^T - \lambda^T\mathbf{N}\ \ (\lambda^T = \mathbf{c}_B^T\mathbf{B}^{-1})$

   If $\bar{\mathbf{c}}_N^T \geq 0$, the optimal solution is obtained! (STOP)
2. Determine which $\mathbf{a}_q$ is to enter the basis by selecting the most negative *reduced cost coeff*. and calculate the column to be pivoted $\mathbf{y}_q = \mathbf{B}^{-1}\mathbf{a}_q$.
3. If no $y_{iq} > 0$, stop! (*The problem is unbounded*.) Otherwise, calculate x_{Bi}/y_{iq} for $y_{iq} > 0$ to determine which vector is to leave the basis by *minimum ratio rule*.
4. Update $\mathbf{B}^{-1}$ and $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, then return to step 1.

Example:

$$\max Z = 3x_1 + x_2 + 3x_3$$

$$\text{subject to}\ \begin{cases} 2x_1 + x_2 + x_3 \leq 2 \\ x_1 + 2x_2 + 3x_3 \leq 5 \\ 2x_1 + 2x_2 + x_3 \leq 6 \end{cases} \qquad x_i \geq 0 \quad (i = 1, 2, 3)$$

The original tableau:

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$	**b**
	2	1	1	1	0	0	2
	1	2	3	0	1	0	5
	2	2	1	0	0	1	6
$\mathbf{c}^T$	-3	-1	-3	0	0	0	0

Start with initial basic feasible solution and corresponding $\mathbf{B}^{-1}$.

var	$\mathbf{B}^{-1}$			$\mathbf{x}_B$	$\mathbf{y}_q$	x_{Bi}/y_{iq}
4	1	0	0	2	(2)	2/2
5	0	1	0	5	1	5/1
6	0	0	1	6	2	6/2

$\bar{\mathbf{c}}_N^T = \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1} = [-3\ \ -1\ \ -3] - [0\ 0\ 0]\mathbf{I}$ (nonbasic index: $\mathbf{I}_N$=[1 2 3])

Update the tableau by pivoting with new basis $\mathbf{y}_1=\mathbf{B}^{-1}\mathbf{a}_1$=[2 1 2].

var	$\mathbf{B}^{-1}$			$\mathbf{x}_B$	$\mathbf{y}_q$	x_{Bi}/y_{iq}
1	1/2	0	0	1	2→1	
5	-1/2	1	0	4	1→0	
6	-1	0	1	4	2→0	

$\bar{\mathbf{c}}_N^T = [-1\ \ -3\ \ 0] - [-3\ 0\ 0]\mathbf{B}^{-1} = [1/2\ \ -3\ \ 0]$ (nonbasic index: $\mathbf{I}_N$=[2 3 4])

Update the tableau with new basis $\mathbf{y}_3=\mathbf{B}^{-1}\mathbf{a}_3$=[1/2 5/2 0].

var	$\mathbf{B}^{-1}$			$\mathbf{x}_B$	$\mathbf{y}_q$	x_{Bi}/y_{iq}
1	1/2	0	0	1	1/2	2/1
3	-1/2	1	0	4	(5/2)	8/5
6	-1	0	1	4	0	big

After pivoting,

var	$\mathbf{B}^{-1}$			$\mathbf{x}_B$	$\mathbf{y}_q$	x_{Bi}/y_{iq}
1	3/5	-1/5	0	1/5	1/2→0	
3	-1/5	2/5	0	8/5	5/2→1	
6	-1	0	1	4	0→0	

$\bar{\mathbf{c}}_N^T = [-1\ 0\ 0] - [-3\ \ -3\ \ 0]\mathbf{B}^{-1} = [1/5\ \ 3/5\ \ 0]$ (nonbasic index: $\mathbf{I}_N$=[2 4 5])

No nonpositive relative cost! ⇒ The optimal solution is obtained.

x_1=1/5, x_2=0, x_3=8/5, x_4=0, x_5=0, x_6=4 and Z= –3x1/5–3x8/5= –27/5.

<Modifications>

- Not to choose *most negative reduced cost* in step 2, but to choose *first negative cost*
  → more major iterations but less computing time in total.
- If we start with **I** as $\mathbf{B}^{-1}$ (basis), then the tableau of k-th iteration can be expressed as **ET** where **E** is an ERO's matrix involves all the pivoting operations and **T** is the initial tableau.

  $\mathbf{B}^{-1} = \mathbf{ET} = \mathbf{E}_k\,\mathbf{E}_{k-1} \cdots \mathbf{E}_2\,\mathbf{E}_1\,\mathbf{I}$

  in Step 1, $\lambda^T = \mathbf{c}_B^T\mathbf{B}^{-1} = \mathbf{c}_B^T\mathbf{E}_k\,\mathbf{E}_{k-1} \cdots \mathbf{E}_2\,\mathbf{E}_1$

  in Step 2, $\mathbf{y}_q = \mathbf{B}^{-1}\mathbf{a}_q = \mathbf{E}_k\,\mathbf{E}_{k-1} \cdots \mathbf{E}_2\,\mathbf{E}_1\,\mathbf{a}_q$

  in Step 4, $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} = \mathbf{E}_k\,\mathbf{E}_{k-1} \cdots \mathbf{E}_2\,\mathbf{E}_1\,\mathbf{b}$

  ⇒ The $\mathbf{B}^{-1}$ can be incorporated in the tableau and updated by the recursive form : $(\mathbf{B}^{-1})_k = \mathbf{E}_k\,(\mathbf{B}^{-1})_{k-1}$
- It is not sensible to store whole **E**. It requires only to know $\mathbf{y}_q$ and p to reconstruct **E**.

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & \cdots & -y_{1q}/y_{pq} & \cdots & 0 \\ 0 & 1 & \cdots & -y_{2q}/y_{pq} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/y_{pq} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -y_{mq}/y_{pq} & \cdots & 1 \end{bmatrix} \quad \text{(mxm matrix)}$$

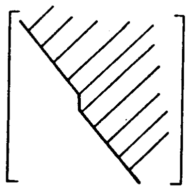
- If the iteration goes on, there could be some accumulation of *truncation error*.
  ⇒ Check $\mathbf{Bx}_B - \mathbf{b}$ periodically and if it is not near zero, then calculate the inversion of **B** and set $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$. (*reinversion*)

- Revised simplex method can be reformulated using **B** instead of $\mathbf{B}^{-1}$.
  - ⇒ Instead of using $\mathbf{B}^{-1}$, solve linear equations three times.

    $\mathbf{B}\mathbf{y}_0 = \mathbf{b}$, $\lambda^T\mathbf{B} = \mathbf{c}_B^T$, $\mathbf{B}\mathbf{y}_q = \mathbf{a}_q$ for $\mathbf{y}_0$, λ, $\mathbf{y}_q$

$$\mathbf{B} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \end{bmatrix} = \mathbf{LU} \quad \text{(initially obtained)}$$

where $\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{bmatrix}$ (upper triangular)

$\bar{\mathbf{H}} =$ 

Let $\bar{\mathbf{B}}$ is the *new basis* where a column $\mathbf{a}_k$ is replaced with $\mathbf{a}_q$.

$$\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{k-1} & \mathbf{a}_{k+1} & \cdots & \mathbf{a}_m & \mathbf{a}_q \end{bmatrix} = \bar{\mathbf{L}}\bar{\mathbf{U}}$$

$$\bar{\mathbf{H}} = \mathbf{L}^{-1}\bar{\mathbf{B}} = \begin{bmatrix} \mathbf{L}^{-1}\mathbf{a}_1 & \cdots & \mathbf{L}^{-1}\mathbf{a}_{k-1} & \mathbf{L}^{-1}\mathbf{a}_{k+1} & \cdots & \mathbf{L}^{-1}\mathbf{a}_m & \mathbf{L}^{-1}\mathbf{a}_q \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_{k-1} & \mathbf{u}_{k+1} & \cdots & \mathbf{u}_m & \mathbf{L}^{-1}\mathbf{a}_q \end{bmatrix}$$

(Non-upper triangular! It has some subdiagonals after (k-1)th column.)

⇒ $\bar{\mathbf{H}}$ can be constructed without additional computation, since $\mathbf{u}_i$'s are known and $\mathbf{L}^{-1}\mathbf{a}_q$ is a by-product in the computation of $\mathbf{y}_q$.

cf) Solving linear equation by LU decomposition

i) **Ax**=**b**=**LUx** ⇒ **L**(**Ux**)=**Ly**=**b**: Since **L** is a *lower triangular matrix*,

$y_1=b_1/l_{11}$,

$y_2=(b_2-l_{21}\,y_1)/l_{22}$,

$y_3=(b_3-l_{31}\,y_1-l_{32}\,y_2)/l_{33}$, and so on.

ii) **Ux**=**y**: Since **U** is a *upper triangular matrix*,

$x_n=y_n/u_{nn}=y_n$,

$x_{n-1}=(y_{n-1}-u_{(n-1)n}\,x_n)/\,u_{(n-1)\,(n-1)}$,

$x_{n-2}=(y_{n-2}-u_{(n-2)\,(n-1)}\,x_{n-1}-u_{(n-2)n}\,x_n)/\,u_{(n-2)\,(n-2)}$, and so on.

Reduction of $\bar{\mathbf{H}}$ to upper diagonal matrix:

$$\mathbf{M}_i = \begin{bmatrix} 1 & 0 & \cdots & & & & 0 \\ 0 & 1 & 0 & & & & 0 \\ \vdots & 0 & \ddots & \ddots & & & \vdots \\ & & \ddots & 1 & 0 & & \\ & & & m_i & 1 & \ddots & \\ & & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & & & 0 & 1 \end{bmatrix} \quad \text{for } i = k, k+1, \cdots, m-1$$

where m_i = obtained from the Gaussian elimination to convert non-upper triangular matrix to triangular matrix form. ($m_i = -h_{(i+1)i} / h_{ii}$)

$\bar{\mathbf{U}} = \mathbf{M}_{m-1}\mathbf{M}_{m-2}\cdots\mathbf{M}_k\bar{\mathbf{H}}$ (*upper triangular matrix with unit diagonals*)

$$\bar{\mathbf{B}} = \mathbf{L}\bar{\mathbf{H}} = \mathbf{L}\mathbf{M}_k^{-1}\mathbf{M}_{k+1}^{-1}\cdots\mathbf{M}_{m-1}^{-1}\bar{\mathbf{U}}$$

$$\bar{\mathbf{L}} = \mathbf{L}\mathbf{M}_k^{-1}\mathbf{M}_{k+1}^{-1}\cdots\mathbf{M}_{m-1}^{-1}$$

($\mathbf{M}_i^{-1}$ is simply $\mathbf{M}_i$ with the sign of the off-diagonal term reversed.)

cf) For the sake of storage convenience, **U** will be decomposed as a upper diagonal matrix with unit diagonals in LU decomposition.

V. DANTZIG-WOLFE DECOMPOSITION METHOD

$$\min_{\mathbf{x}} Z = \mathbf{c}^T \mathbf{x}$$

Subject to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$

- If **A** has the special block-angular structure,

$$\mathbf{A} = \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 & \cdots & \mathbf{L}_N \\ \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_N \end{bmatrix}$$

It becomes

$$\min_{\mathbf{x}} \sum_{i=1}^{N} \mathbf{c}_i^T \mathbf{x}_i$$

Subject to $\sum_{i=1}^{N} \mathbf{L}_i \mathbf{x}_i = \mathbf{b}_0$,

$\mathbf{A}_i \mathbf{x}_i = \mathbf{b}_i$ and $\mathbf{x}_i \geq \mathbf{0}$ $\quad (i = 1, 2, \cdots, N)$

⇒ Divisions into N subproblems with a linking constraint of dimension m.

Let the constraint set for the i-th subproblem be $\mathbf{S}_i = \{\mathbf{x}_i : \mathbf{A}_i\mathbf{x}_i = \mathbf{b}_i, \mathbf{x}_i \geq \mathbf{0}\}$ and assume that each of the polytopes $\mathbf{S}_i$ ($i = 1, \cdots, N$) is indeed bounded and hence a polyhedron (by placing artificial large upper bounds on each $\mathbf{x}_i$). And let the extreme points of $\mathbf{S}_i$ be $\{\mathbf{x}_{i1}, \mathbf{x}_{i2}, \cdots, \mathbf{x}_{iK_i}\}$.

Then, $\mathbf{x}_i = \sum_{j=1}^{K_i} \alpha_{ij} \mathbf{x}_{ij}$ where $\sum_{j=1}^{K_i} \alpha_{ij} = 1$ and $\alpha_{ij} \geq 0$ for j=1, …K_i (linear combination) and the original problem becomes:

$$Z = \sum_{i=1}^{N} \mathbf{c}_i^T \mathbf{x}_i = \sum_{i=1}^{N} \mathbf{c}_i^T \sum_{j=1}^{K_i} \alpha_{ij} \mathbf{x}_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{K_i} \mathbf{c}_i^T \mathbf{x}_{ij} \alpha_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{K_i} p_{ij} \alpha_{ij} \quad (p_{ij} = \mathbf{c}_i^T \mathbf{x}_{ij})$$

$$\sum_{i=1}^{N} \mathbf{L}_i \mathbf{x}_i = \sum_{i=1}^{N} \mathbf{L}_i \sum_{j=1}^{K_i} \alpha_{ij} \mathbf{x}_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{K_i} \mathbf{L}_i \mathbf{x}_{ij} \alpha_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{K_i} \mathbf{q}_{ij} \alpha_{ij} = \mathbf{b}_0 \quad (\mathbf{q}_{ij} = \mathbf{L}_i \mathbf{x}_{ij})$$

<Master problem>

$$\min_{\boldsymbol{\alpha}} \mathbf{p}^T \boldsymbol{\alpha}$$

Subject to $\mathbf{Q}\boldsymbol{\alpha} = \mathbf{g}$ and $\boldsymbol{\alpha} \geq \mathbf{0}$

where $\boldsymbol{\alpha} = [\alpha_{11} \cdots \alpha_{1K_1} \alpha_{21} \cdots \alpha_{2K_1} \cdots \alpha_{NK_N}]^T$, $\boldsymbol{g} = [\mathbf{b}_0^T, 1, 1, \cdots, 1]^T$,

$$\mathbf{p}^T = [\mathbf{c}_1^T \mathbf{x}_{11} \cdots \mathbf{c}_1^T \mathbf{x}_{1K_1} \mathbf{c}_2^T \mathbf{x}_{21} \cdots \mathbf{c}_2^T \mathbf{x}_{2K_2} \cdots \mathbf{c}_N^T \mathbf{x}_{NK_N}]$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}_{11} & \cdots & \mathbf{q}_{1K_1} & \mathbf{q}_{21} & \cdots & \mathbf{q}_{2K_2} & \cdots & \mathbf{q}_{NK_N} \\ 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

Minimum relative cost: $r^* = \min_{i\in[1,\cdots,N]} \left\{ r_i^* = \min_{j\in[1,\cdots,K_i]} \left(p_{ij} - \left[\mathbf{p}_B^T \mathbf{B}^{-1} \mathbf{N} \right]_j \right) \right\}$ (with $\mathbf{p}_B^T\mathbf{B}^{-1} \to \lambda^T$)

This procedure will calculate even for the relative costs of basic variables that will be zero. But it does not affect the results of the procedure.

$$r_i^* = \min_{j\in[1,\cdots,K_i]} \left(\mathbf{c}_i^T \mathbf{x}_{ij} - \left[\boldsymbol{\lambda}_0^T \quad \lambda_{m+1} \right] \begin{bmatrix} \mathbf{q}_{ij} \\ \mathbf{e}_i \end{bmatrix} \right) = \min_{j\in[1,\cdots,K_i]} \left(\mathbf{c}_i^T \mathbf{x}_{ij} - \boldsymbol{\lambda}_0^T \mathbf{L}_i \mathbf{x}_{ij} - \lambda_{m+1} \right)$$

where $\boldsymbol{\lambda}_0$ is the vector made up of first m elements of $\boldsymbol{\lambda}$. (The $\mathbf{x}_{ij}$'s are in $\mathbf{S}_i$ and they satisfy $\mathbf{A}\mathbf{x}_i = \mathbf{b}_i$.)

<The *i*-th subproblem>

$$\min_{\mathbf{x}_i} (\mathbf{c}_i^T - \boldsymbol{\lambda}_0^T \mathbf{L}_i)\mathbf{x}_i$$

$$\text{Subject to } \mathbf{A}_i\mathbf{x}_i = \mathbf{b}_i \text{ and } \mathbf{x}_i \geq \mathbf{0}$$

⇒ Solve the *i*-th subproblem to get $\mathbf{x}_i^*$ and calculate r_i^* in the procedure of solving master problem.

Example:

$$\min Z = -x_1 - 2x_2 - 4x_3 - 3x_4$$

Subject to

$$x_1 + x_2 + 2x_3 \leq 4$$
$$x_2 + x_3 + x_4 \leq 3$$
$$2x_1 + x_2 \leq 4 \quad \text{and} \quad x_i \geq 0 \ (i = 1,2,3,4)$$
$$x_1 + x_2 \leq 2$$
$$x_3 + x_4 \leq 2$$
$$3x_3 + 2x_4 \leq 5$$

(In the first two constraints, the x_1, x_2 part is marked $\mathbf{L}_1$ and the x_3, x_4 part $\mathbf{L}_2$; the constraints in x_1, x_2 form $\mathbf{A}_1$ and those in x_3, x_4 form $\mathbf{A}_2$.)

Slack variables will be added, but the number of decision variables in the master problem will be same as those of the original problem.

<*Master problem*: 6 variables and 4 constraints>

$$\min_{\alpha} (p_{11}\alpha_{11} + p_{12}\alpha_{12} + p_{21}\alpha_{21} + p_{22}\alpha_{22})$$

$$\text{Subject to } \alpha_{11}\mathbf{L}_1\mathbf{x}_{11} + \alpha_{12}\mathbf{L}_1\mathbf{x}_{12} + \alpha_{21}\mathbf{L}_2\mathbf{x}_{21} + \alpha_{22}\mathbf{L}_2\mathbf{x}_{22} + \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\alpha_{11} + \alpha_{12} = 1, \quad \alpha_{21} + \alpha_{22} = 1$$

$$\text{and } \alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}, s_1, s_2 \geq 0$$

where $p_{11} = [-1\ -2]\mathbf{x}_{11}$, $p_{12} = [-1\ -2]\mathbf{x}_{12}$, $p_{21} = [-4\ -3]\mathbf{x}_{21}$, $p_{22} = [-4\ -3]\mathbf{x}_{22}$.

A starting basic feasible solution can be $[s_1, s_2, \alpha_{11}, \alpha_{21}] = [4\ 3\ 1\ 1]$ (Thus, it requires only $\mathbf{x}_{11}$ and $\mathbf{x}_{21}$ for revised simplex method.) and the convenient extreme points of the subsystems are $\mathbf{x}_{11}=\mathbf{0}$ and $\mathbf{x}_{21}=\mathbf{0}$. To select the nonbasic variable to be basic variable, solve the two subproblems.

$$\min (\mathbf{c}_1^T - \lambda_0^T \mathbf{L}_1)\mathbf{x}_1 \qquad\qquad \min (\mathbf{c}_2^T - \lambda_0^T \mathbf{L}_2)\mathbf{x}_2$$

$$\text{s.t } \begin{aligned} 2x_1 + x_2 &\leq 4 \\ x_1 + x_2 &\leq 2 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned} \quad \text{and} \quad \text{s.t } \begin{aligned} x_3 + x_4 &\leq 2 \\ 3x_3 + x_4 &\leq 5 \\ x_3 \geq 0, x_4 &\geq 0 \end{aligned}$$

where $\mathbf{c}_1^T = [-1 \quad -2]$, $\mathbf{c}_2^T = [-4 \quad -3]$, and $\lambda_0 = [0 \quad 0]$. Then, find the minimum relative price among two subproblem solutions ($\mathbf{x}_1$=[0 2]^T, $\mathbf{x}_2$=[1 1]^T and r_1=([–1 –2] –$\mathbf{0}$)$\mathbf{x}_1$–0= –4, r_2=([–4 –3] –$\mathbf{0}$)$\mathbf{x}_2$–0= –7) in order to get the nonbasic variable to enter as basic (α_{22} for $\mathbf{x}_{22}$=[1 1]^T) and find the basic variable to be nonbasic using minimum ratio rule (α_{21}). Iterate the procedure until optimum is obtained.

VI. DUALITY

<u>**Primal LP**</u>	<u>**Dual LP**</u>
min $\mathbf{c}^T\mathbf{x}$	max $\lambda^T\mathbf{b}$
s. t. $\mathbf{Ax} \geq \mathbf{b}$	s. t. $\lambda^T\mathbf{A} \leq \mathbf{c}^T$
$\mathbf{x} \geq \mathbf{0}$	$\lambda \geq \mathbf{0}$

(*Symmetric form*)

- Consider a standard LP problem

$$\begin{aligned}&\min \mathbf{c}^T\mathbf{x}\\ &\text{s. t. } \mathbf{Ax} = \mathbf{b}\\ &\quad \mathbf{x} \geq \mathbf{0}\end{aligned} \quad \Leftrightarrow \quad \begin{aligned}&\min \mathbf{c}^T\mathbf{x}\\ &\text{s. t. } \mathbf{Ax} \geq \mathbf{b}\\ &\quad \mathbf{Ax} \leq \mathbf{b}\\ &\quad \mathbf{x} \geq \mathbf{0}\end{aligned} \quad \Leftrightarrow \quad \begin{aligned}&\min \mathbf{c}^T\mathbf{x}\\ &\text{s. t. } \begin{bmatrix}\mathbf{A}\\ -\mathbf{A}\end{bmatrix}\mathbf{x} \geq \begin{bmatrix}\mathbf{b}\\ -\mathbf{b}\end{bmatrix}\\ &\quad \mathbf{x} \geq \mathbf{0}\end{aligned}$$

Its dual is:

$$\begin{aligned}&\max(\mathbf{u}^T\mathbf{b} - \mathbf{v}^T\mathbf{b})\\ &\text{s. t. } \mathbf{u}^T\mathbf{A} - \mathbf{v}^T\mathbf{A} \leq \mathbf{c}^T\\ &\quad \mathbf{u} \geq \mathbf{0},\ \ \mathbf{v} \geq \mathbf{0}\end{aligned} \quad \xrightarrow{\lambda = \mathbf{u}-\mathbf{v}} \quad \begin{aligned}&\max \lambda^T\mathbf{b}\\ &\text{s. t. } \lambda^T\mathbf{A} \leq \mathbf{c}^T\\ &(\lambda \text{ is free})\end{aligned}$$

(*Asymmetric form*)

Lemma 1: *Weak Duality Theorem*

If $\mathbf{x}$ and λ are feasible for the asymmetric primal and its dual problems, respectively, then $\mathbf{c}^T\mathbf{x} \geq \lambda^T\mathbf{b}$.

pf) $\lambda^T\mathbf{A} \leq \mathbf{c}^T$, $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{Ax=b}$ $\Rightarrow$ $\mathbf{c}^T\mathbf{x} \geq \lambda^T\mathbf{Ax} = \lambda^T\mathbf{b}$.

Remark: A feasible vector to either problem yields a *bound* on the objective function value of the other problem.

Corollary:

If x_0 and λ_0 are feasible for the asymmetric primal and its dual problems, respectively, and if $\mathbf{c}^T\mathbf{x}_0 = \lambda^T\mathbf{b}_0$, then $\mathbf{x}_0$ and λ_0 are optimal for their respective problems.

Theorem: *Duality Theorem of LP*

If either of the asymmetric primal and its dual problems has a finite optimal solution, so does the other, and the corresponding values of the objective functions are equal. If either problem has an unbounded objective, the other is not feasible.

- Relations between the primal and dual problems

Let $\mathbf{A}$=[$\mathbf{B}$ $\mathbf{N}$]. If a basic feasible solution $\mathbf{x}_B$=$\mathbf{B}^{-1}\mathbf{b}$ is optimal, the relative cost vector $\mathbf{r}$ must be nonnegative in each component.

$$\mathbf{r}_N^T = \mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N} \geq 0 \quad \Rightarrow \quad \mathbf{c}_N^T \geq \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N}$$

Let $\boldsymbol{\lambda}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$ (*simplex multiplier*). Then, at the optimal solution

$$\boldsymbol{\lambda}^T \mathbf{A} = \begin{bmatrix} \boldsymbol{\lambda}^T \mathbf{B} & \boldsymbol{\lambda}^T \mathbf{N} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B^T & \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N} \end{bmatrix} \leq \begin{bmatrix} \mathbf{c}_B^T & \mathbf{c}_N^T \end{bmatrix} = \mathbf{c}^T$$

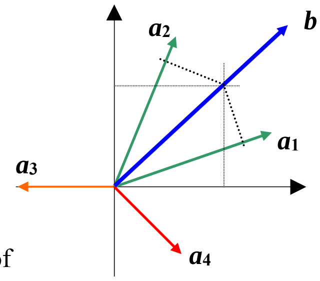
$$\boldsymbol{\lambda}^T \mathbf{b} = \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{b} = \mathbf{c}_B^T \mathbf{x}_B = \mathbf{c}^T \mathbf{x}$$

Theorem: Alternative *Duality Theorem of LP*

Let the LP of the asymmetric primal problem have an optimal basic feasible solution corresponding to the basis **B**. Then the vector $\boldsymbol{\lambda}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$ is an optimal solution to its dual problem and the optimal values of both problems are equal.

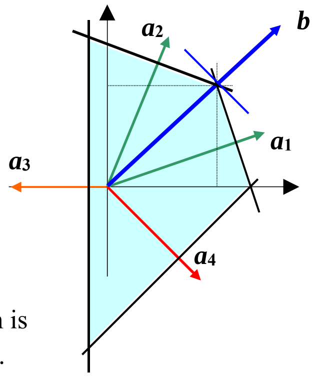
- Geometric interpretation

$$\begin{aligned} &\min Z = 18x_1 + 12x_2 + 2x_3 + 6x_4 \\ &\text{s. t.} \quad 3x_1 + x_2 - 2x_3 + x_4 = 2 \\ &\qquad x_1 + 3x_2 \qquad - x_4 = 2 \\ &\qquad x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0 \end{aligned}$$



In primal space, find a positive linear combination of $\mathbf{a}_i$'s to yield resource vector **b**. (unique in this case)

$$\begin{aligned} &\max Z = 2\lambda_1 + 2\lambda_2 \\ &\text{s. t.} \quad 3\lambda_1 + \lambda_2 \leq 18 \\ &\qquad \lambda_1 + 3\lambda_2 \leq 12 \\ &\qquad -2\lambda_1 \qquad \leq 2 \\ &\qquad \lambda_1 - \lambda_2 \leq 6 \end{aligned}$$



In dual space, the dual feasible region is determined by the orthogonal lines to each $\mathbf{a}_i$'s of which location is determined by the elements of the resource vector $\mathbf{c}^T$.

- *Simplex Multipliers*

$$\boldsymbol{\lambda}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$$

- ▶ The λ_j is a synthetic price as a linear combination of the original costs.
- ▶ This vector is not a solution to dual problem unless **B** is an optimal basis for primal.
- ▶ Nevertheless, it has economic interpretation (related to relative cost, *shadow price*).
- ▶ If the primal problem is to produce m-products **b** at the minimum cost of $\mathbf{c}^T\mathbf{x}$ by n-facilities **x** with the production rate of each product **A**, then the dual problem is to maximize the product purchase ($\boldsymbol{\lambda}^T\mathbf{b}$) not by manufacturing while the purchase price ($\boldsymbol{\lambda}^T\mathbf{a}_i$) at the same production rate of each facility should be less than the production cost ($\boldsymbol{\lambda}^T\mathbf{a}_i \leq c_i$ or $\boldsymbol{\lambda}^T\mathbf{A} \leq \mathbf{c}^T$).

- *Complementary slackness*

1. Asymmetric case (standard LP form, $\mathbf{Ax} = \mathbf{b}$):

From the duality theorem, $\boldsymbol{\lambda}^T\mathbf{b} = \mathbf{c}^T\mathbf{x} \quad \Rightarrow \quad (\boldsymbol{\lambda}^T\mathbf{A} - \mathbf{c}^T)\mathbf{x} = \mathbf{0}$

$x_i > 0$ implies $\boldsymbol{\lambda}^T\mathbf{a}_i = c_i$ ($s_i = 0$)

$\boldsymbol{\lambda}^T\mathbf{a}_i < c_i$ implies $x_i = 0$ ($s_i > 0$).

2. Symmetric case (nonstandard LP form, $\mathbf{Ax} \geq \mathbf{b}$):

From the duality theorem, $\lambda^T\mathbf{b} = \mathbf{c}^T\mathbf{x} \;\Rightarrow\; (\lambda^T\mathbf{A} - \mathbf{c}^T)\mathbf{x} \geq \mathbf{0}$

$x_i > \mathbf{0}$ implies $\lambda^T\mathbf{a}_i \geq c_i \;\Rightarrow\; \lambda^T\mathbf{a}_i = c_i \; (s_{\lambda i} = 0)$

$\lambda^T\mathbf{a}_i < \mathbf{c}_i$ implies $x_i \leq 0. \;\Rightarrow\; x_i = 0 \; (s_{xi} = 0)$

From the duality theorem, $\lambda^T\mathbf{b} = \mathbf{c}^T\mathbf{x} \;\Rightarrow\; \lambda^T(\mathbf{Ax} - \mathbf{b}) \leq \mathbf{0}$

$\lambda_j > \mathbf{0}$ implies $\mathbf{a}^j\mathbf{x} \leq b_j \;\Rightarrow\; \mathbf{a}^j\mathbf{x} = b_j \; (s_{xi} = 0)$

$\mathbf{a}^j\mathbf{x} > b_i$ implies $\lambda_j \leq \mathbf{0} \;\Rightarrow\; \lambda_j = \mathbf{0}$.

(where $\mathbf{a}^j$ is the j-th row of $\mathbf{A}$)

- Physical meaning of λ in terms of *sensitivity*

Let $\mathbf{x} = [\mathbf{x}_B \; \mathbf{0}]^T$ be the optimal basic feasible solution and $\boldsymbol{B}$ be the corresponding optimal basis. We know that $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ and $\lambda^T = \mathbf{c}_B{}^T\mathbf{B}^{-1}$. Assume the requirement vector $\mathbf{b}$ is changed to $\mathbf{b} + \Delta\mathbf{b}$. The optimal solution is then $[\mathbf{x}_B + \Delta\mathbf{x}_B \; \mathbf{0}]^T$ where $\Delta\mathbf{x}_B = \mathbf{B}^{-1}\Delta\mathbf{b}$.

The corresponding increment in the cost objective will be

$\Delta Z = \mathbf{c}_B{}^T\Delta\mathbf{x}_B = \mathbf{c}_B{}^T\mathbf{B}^{-1}\Delta\mathbf{b} = \lambda^T\Delta\mathbf{b}$

or $\lambda_i = \Delta z / \Delta b_i$

$\Rightarrow$ Sensitivity of the optimal cost with respect to $\mathbf{b}$. (*Marginal price*)

VII. SENSITIVITY (POST-OPTIMALITY) ANALYSIS

- By changing input coefficients (resource or constraints coefficients), the optimal value can be improved considerably → Then, it should be considered to change the situation.
- Decide the importance of the data coefficients which enables to re-estimate the important data coefficients → Improve the accuracy, reliability of the model.

▶ *Ranging of the Coefficients*

- Objective *function coefficient* (c_i): within some ranges of each coefficient, the *optimal solution* will not change even though the optimal value will change by $x_i^*\Delta_i$ (the slope of the objective function will change).

$$\tilde{\mathbf{r}}_N^T = \mathbf{c}_N^T - \tilde{\mathbf{c}}_B^T\mathbf{B}^{-1}\mathbf{N} \quad \text{where } \tilde{c}_{Bi} = c_{Bi} + \Delta_i \quad (\Delta_j = 0 \text{ for } j \neq i)$$

For the optimal solution to be same, the new relative costs for nonbasic variable should be nonnegative despite the change in cost coefficient c_i.

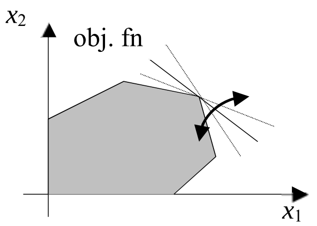
$$\tilde{r}_{Nj} = r_{Nj} - \Delta_i(\mathbf{B}^{-1}\mathbf{N})_{ij} \geq 0, \; \forall j \quad \text{(Minimization)}$$

- Resource *coefficient* (b_i): within some ranges of each resource coefficient, the *optimal mix* will not change even though the optimal solution and value will change ($\lambda_i^*\Delta_i$).

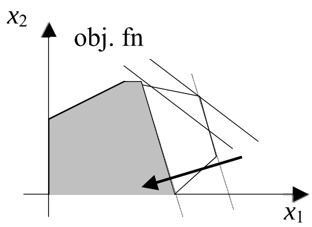
$$\tilde{\mathbf{x}}_B^* = \mathbf{B}^{-1}\tilde{\mathbf{b}} \quad \text{where } \tilde{b}_i = b_i + \Delta_i \quad (\Delta_j = 0 \text{ for } j \neq i)$$

For the optimal mix to be same, the new optimal solution should be positive despite the change in cost coefficient b_i.

$$\tilde{x}_{Bj} = x_{Bj} + (\mathbf{B}^{-1})_{ji}\Delta_i > 0, \; \forall j$$



(no change in obj. fn value)



(x_2 will not exist if it goes further)

▶ *Simultaneous Variations in Parameters*

- ***100% rule*** **(For objective function coefficients)**

Let δc_j be the actual decrease (increase) in the objective function coefficient of variable x_j and Δc_j be the maximum decrease (increase) allowed by the sensitivity analysis.

If $\sum_j \frac{\delta c_j}{\Delta c_j} \le 1$ satisfies, *the optimal solution* will not change. The change in objective function value will be $\Delta Z = \sum_j \delta c_j x_j^*$.

Remark: The failure of the 100% rule for objective function coefficients does not imply that the optimal solution will change.

- ***100% rule*** **(For resource coefficient)**

Let δb_i be the actual decrease (increase) in the resource coefficient of the i-th constraint and Δb_i be the maximum decrease (increase) allowed by the sensitivity analysis.

If $\sum_i \frac{\delta b_i}{\Delta b_i} \le 1$ satisfies, *the optimal product mix* and *the shadow prices* will not change.

The change in objective function value will be $\Delta Z = \sum_i \delta b_i \lambda_i^*$.

▶ *Adding more variables*

Suppose x_{n+1} is added with constraint coefficient vector $\mathbf{a}_{n+1}$ and the cost c_{n+1}.

$$\tilde{\mathbf{r}}_N^T = \tilde{\mathbf{c}}_N^T - \boldsymbol{\lambda}^T[\mathbf{N}\ \mathbf{a}_{n+1}] \Rightarrow r_{n+1} = c_{n+1} - \boldsymbol{\lambda}^T \mathbf{a}_{n+1} \quad \text{where} \quad \tilde{\mathbf{c}}_N = [\mathbf{c}_N; c_{n+1}].$$

If the r_{n+1} is nonnegative, the x_{n+1} should remain as a nonbasic variable that is zero. Thus,

$$r_{n+1} < 0 \Rightarrow c_{n+1} < \boldsymbol{\lambda}^T \mathbf{a}_{n+1}$$

Then, the new variable can be a basic variable that can have nonzero value and adding a new variable is meaningful.

Example:

A factory can produce four products denoted by P_1, P_2, P_3 and P_4. Each product must be produced in each of two workshops. The processing time (in hours per unit produced) are given in the following table. 400 hours of labor are available in each workshop. The profit margins are 4, 6, 10 and 9 dollars per unit of P_1, P_2, P_3 and P_4 produced, respectively. Everything produced can be sold. Thus, the maximizing profit, the following linear program can be used.

	P_1	P_2	P_3	P_4
Workshop 1	3	4	8	6
Workshop 2	6	2	5	8

Max $4x_1+6x_2+10x_3+9x_4$

Subject to $3x_1+4x_2+8x_3+6x_4 \le 400$

$6x_1+2x_2+5x_3+8x_4 \le 400$

$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0$

Introducing slack variables s_1 and s_2, and applying the simplex method, we get the final tableu:

	x_1	x_2	x_3	x_4	s_1	s_2	b
	0.75	1	2	1.5	0.25	0	100
	4.5	0	1	5	–0.5	1	200
$\mathbf{r}^T$	0.5	0	2	0	1.5	0	600

(a) How many units of P_1, P_2, P_3 and P_4 should be produced in order to maximize profits?

(b) Assume that 20 units of P_3 have been produced by mistake. What is the resulting decrease in profit?

(c) In what range can the profit margin per unit of P_1 vary without changing the optimal basis?

(d) In what range can the profit margin per unit of P_2 vary without changing the optimal basis?

(e) What is the marginal value of increasing the production capacity of Workshop 1?

(f) In what range can the capacity of Workshop 1 vary without changing the optimal basis?

(g) Management is considering the production of a new product P_5 that would require 2 hours in Workshop 1 and 10 hours in Workshop 2. What is the minimum profit margin needed on this new product to make it worth producing?

Answers:

(a) From the final tableau, we read that x_2=100 is basic and x_1= x_3= x_4=0 are nonbasic. So 100 units of P_2 should be produced and none of P_1, P_3 and P_4. The resuting profit is \$600 and that is the maximum possible, given the constraints.

(b) The reduced cost for x_3 is 2 (found in Row 3 of the final tableau). Thus, the effect on profit of producing x_3 units of P_3 is $-2x_3$. If 20 units of P_3 have been produced by mistake, then the profit will be 2x20=\$40 lower than the maximum stated in (a).
(making the coefficient for x_3 one and replace b_2=20 and perform ERO for r_3 to be zero)

(c) Let $4+\Delta$ be the profit margin on P_1. The reduced cost remains nonnegative in the final tableau if $0.5-\Delta \geq 0$ since x_1 is *nonbasic*. That is $\Delta \leq 0.5$. Therefore, as long as the profit margin on P_1 is less than 4.5, the optimal basis remains unchanged.

(d) Let $6+\Delta$ be the profit margin on P_2. Since x_2 is *basic*, we need to restore a correct basis. This is done by adding Δ times Row 1 to Row 3. This effects the reduced costs of the nonbasic variables, namely x_1, x_3, x_4 and s_1. All these reduced costs must be nonnegative. This implies: (sign change in relative cost for maximization)

$$\tilde{r}_{Nj} = r_{Nj} + \Delta(\mathbf{B}^{-1}\mathbf{N})_{1j} \geq 0, \ \forall j$$ (Since x_2 is the first basic variable)

$0.5+0.75\Delta \geq 0$

$2+2\Delta \geq 0$

$0+1.5\Delta \geq 0$

$1.5+0.25\Delta \geq 0$

Combining all these inequalities, we get $\Delta \geq 0$. So, as long as the profit margin on P_2 is 6 or greater, the optimal basis remains unchanged.

(e) The marginal value of increasing capacity in Workshop 1 is $\lambda_1^* = 1.5$.

$$\lambda^T = \mathbf{c}_B^T\mathbf{B}^{-1} = [6 \quad 0]\begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{4}[6 \quad 0]\begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix} = [1.5 \quad 0]$$

(f) Let $400+\Delta$ be the capacity of Workshop 1. The resulting RHS in the final tableau will be: $100+0.25\Delta$ in Row 1, and $200-0.5\Delta$ in Row 2. The optimal basis remains unchanged as long as these two quantities are nonnegative. Namely, $-400 \leq \Delta \leq 400$. So, the optimal basis remains unchanged as long as the capacity of Workshop 1 is in the range 0 to 800.

($\tilde{x}_{B1} = x_{B1} + (1/4)\Delta_1 > 0$ and $\tilde{x}_{B2} = x_{B2} + (-1/2)\Delta_1 > 0$)

(g) The effect on the optimum profit of producing x_5 units of P_5 would be

$$\lambda^{*T}\mathbf{a}_5 = \lambda_1^*(2) + \lambda_2^*(10) = 1.5(2) + 0(10) = 3 \geq c_5 .$$

If the profit margin on P_5 is sufficient to offset this, then P_5 should be produced. That is, we should produce P_5 if its profit margin is at least \$3.

VIII. OTHERS

▶ Dual Simplex Method

If a certain linear programming is solved, then a new problem is constructed by changing the resource vector, **b**. In this case, the solution may not be feasible to the new problem, but the solution is a basic feasible solution for the dual problem (satisfying $\lambda^T\mathbf{a}_i \le c_i$).

The basis **B** satisfies $\lambda^T = \mathbf{c}_B^T\mathbf{B}^{-1}$. However, $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$, called a *dual feasible solution*, may not satisfy $\mathbf{x}_B \ge \mathbf{0}$ and some x_{Bi}'s are negative.

From the complementary slackness for asymmetric case, $\mathbf{x}_i > \mathbf{0}$ implies $\lambda^T\mathbf{a}_i = c_i$

$\lambda^T\mathbf{a}_j = c_j$ for $j = 1, \ldots, m$

$z_j = \lambda^T\mathbf{a}_j < c_j$ for $j = m+1, \ldots, n$

The new marginal price vector is obtained by the exchange of one variable (i-th). Then $\lambda^T\mathbf{a}_i$ should be less than c_i ($c_i - \varepsilon$, $\varepsilon > 0$) and one of $\lambda^T\mathbf{a}_j$ among $j = m+1, \ldots, n$ should be c_i. Let the i-th row of $\mathbf{B}^{-1}$ be $\mathbf{u}^i$. and y_{ij} be $\mathbf{u}^i\mathbf{a}_j$. Then,

$\lambda^T\mathbf{a}_j = c_j$ for $j = 1, \ldots, m$ ($j \ne i$)

$\lambda^T\mathbf{a}_i = c_i - \varepsilon$

$\lambda^T\mathbf{a}_j = z_j - \varepsilon y_{ij}$ for $j = m+1, \ldots, n$

Therefore, choose ε so that the only one of $(z_j - \varepsilon y_{ij})$'s becomes c_j. Since $z_j < c_j$ and $\varepsilon > 0$, the y_{ij} should be negative.

<Procedure>

1. Given a dual feasible solution $\mathbf{x}_B$, if $\mathbf{x}_B \ge \mathbf{0}$, then it is the optimal! If $\mathbf{x}_B$ is not nonnegative, select an index i such that the i-th component of $\mathbf{x}_B$, $x_{Bi} < 0$.
2. If $y_{ij} = (\mathbf{B}^{-1}\mathbf{N})_{ij} \ge 0$ for $j = 1, 2, \cdots, n$, then the dual has no maximum. If $y_{ij} < 0$ for some j, then let

$$\varepsilon_0 = \frac{z_k - c_k}{y_{ik}} = \min_j \left\{ \frac{z_j - c_j}{y_{ij}} : y_{ij} < 0 \right\}$$

3. Form a new basis **B** by replacing $\mathbf{a}_i$ by $\mathbf{a}_k$. Using this basis determine the corresponding basic dual feasible solution $\mathbf{x}_B$ and return to step 1.

⇒ It does not require an initial basic feasible solution for **x**.

▶ Primal-Dual Algorithm

$$\begin{array}{lcl} \min \mathbf{c}^T\mathbf{x} & & \max \lambda^T\mathbf{b} \\ \text{s. t. } \mathbf{A}\mathbf{x} = \mathbf{b} & \Leftrightarrow & \text{s. t. } \lambda^T\mathbf{A} \le \mathbf{c}^T \\ \quad \mathbf{x} \ge \mathbf{0} & & (\lambda \text{ is free}) \end{array}$$

Given a feasible solution λ to the dual problem, let $P = \{ i \mid i = 1,2,\ldots,m \}$.

$\lambda^T\mathbf{a}_i = c_i, \ \forall i \in P$ (basic)

$\lambda^T\mathbf{a}_i < c_i, \ \forall i \notin P$ (nonbasic)

Associated restricted primal and dual problems:

$$\begin{array}{lcl} \min \mathbf{1}^T\mathbf{y} & & \\ \text{s. t. } \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b} & \Leftrightarrow & \begin{array}{l} \max \mathbf{u}^T\mathbf{b} \\ \text{s. t. } \mathbf{u}^T[\mathbf{A} \ \ \mathbf{I}] \le [\mathbf{0} \ \ \mathbf{1}] \end{array} \\ \quad \mathbf{x}, \mathbf{y} \ge \mathbf{0} & & \end{array}$$

Theorem: (Primal-Dual optimality theorem)

Suppose the λ is feasible for the dual and that $\mathbf{y}=\mathbf{0}$ and $\mathbf{x}_B$ is the optimal for the associated restricted primal. Then $\mathbf{x}=[\mathbf{x}_B;\ \mathbf{0}]$ and λ are optimal for the original primal and dual problem, respectively.

<Procedure>

1. Given a feasible solution λ_0 to the original dual problem, set up the associated restricted primal problem.
2. Optimize the associated restricted primal. If the minimal value of this problem is zero, the corresponding solution is optimal for the original primal problem by the primal-dual optimality theorem.
3. If the minimal value of the associated restricted primal is strictly positive, obtain the solution $\mathbf{u}_0$ of the associated restricted dual from the final simplex of the associated restricted primal. If there is no j for which $\mathbf{u}_0^T\mathbf{a}_j>0$, conclude the primal has no feasible solutions. Else, define the new dual feasible vector $\lambda = \lambda_0+\varepsilon_0\mathbf{u}_0$

   where $\varepsilon_0 = \dfrac{c_k - \lambda_0^T\mathbf{a}_k}{\mathbf{u}_0^T\mathbf{a}_k} = \min_j\left\{\dfrac{c_j - \lambda_0^T\mathbf{a}_j}{\mathbf{u}_0^T\mathbf{a}_j} : \mathbf{u}_0^T\mathbf{a}_j < 0\right\}$

   Then go back to Step 1 using this λ.

▶ **Reduction of Linear Inequalities**

1. *Redundant equations*: Corresponding to the system of linear constraints $\mathbf{Ax}=\mathbf{b}$, $\mathbf{x}\geq\mathbf{0}$, the system is said to have *redundant equations* if there is a nonzero m-vector $\mathbf{w}$ satisfying

   $\mathbf{w}^T\mathbf{A}=\mathbf{0}$ and $\mathbf{w}^T\mathbf{b} = 0$.

   → $rank(\mathbf{A})<m$: imposes unnecessary computation

   → It can be detected and eliminated in Phase I (if the tableau has zero rows).

2. *Null variables*: A *variable* x_i in the system of linear constraints $\mathbf{Ax}=\mathbf{b}$, $\mathbf{x}\geq\mathbf{0}$ is said to be a *null variable* if $x_i = 0$ in every solution.

   → Eliminate the null variables and i-th column of $\mathbf{A}$ from the system.

   Null Variable Theorem: If the feasible region S is not empty, the variable x_i is a *null variable* if and only if there is a nonzero m-vector $\mathbf{w}$ such that $\mathbf{w}^T\mathbf{A}\geq\mathbf{0}$ and $\mathbf{w}^T\mathbf{b} = 0$ and the i-th component of $\mathbf{w}^T\mathbf{A}$ is strictly positive.

   cf) If is a null variable the following LP has the zero optimal value.

   $$\begin{array}{l}\min(-\mathbf{e}^i\mathbf{x})\\ \text{s. t. } \mathbf{Ax}=\mathbf{b}\\ \quad\ \ \mathbf{x}\geq\mathbf{0}\end{array} \quad\Leftrightarrow\quad \begin{array}{l}\max \lambda^T\mathbf{b}\\ \text{s. t. } \lambda^T\mathbf{A}\leq-\mathbf{e}^i\end{array}$$

   Thus, with $\mathbf{w} = -\lambda$, $\mathbf{w}^T\mathbf{b} = 0$, $\mathbf{w}^T\mathbf{A}\geq\mathbf{0}$ and so on.

3. *Nonextremal variables*: A variable x_i in the system of linear constraints $\mathbf{Ax}=\mathbf{b}$, $\mathbf{x}\geq\mathbf{0}$ is *nonextremal* if the inequality $x_i \geq 0$ is redundant.

   → Treat them as free variables and eliminate them by using equations where they are expressed in terms of other variables

   → If the inequality, $x_i \geq 0$ can be composed of a linear combination of the inequalities, $\mathbf{Ax}\geq\mathbf{b}$, then x_i is a nonextremal variable.

Nonextremal Variable Theorem: If the feasible region S is not empty, the variable x_i is a *nonextremal variable* if and only if there is m-vector $\mathbf{w}$ and n-vector $\mathbf{d}$ such that $\mathbf{w}^T\mathbf{A}=\mathbf{d}^T$ and $\mathbf{w}^T\mathbf{b}\le 0$ where $d_j = -1$ and $d_i \ge 0$ ($i \ne j$).

cf) Let $\mathbf{w}^T\mathbf{b} = -\beta \quad (\beta \ge 0)$ and $x_j \ge \beta$.

$$\begin{array}{lcl} \min x_j & & \max \boldsymbol{\lambda}^T\mathbf{b} \\ \text{s. t. } \mathbf{Ax}=\mathbf{b} & \Leftrightarrow & \text{s. t. } \boldsymbol{\lambda}^T\mathbf{a}_i \le 0 \ (i \ne j) \\ \quad x_i \ge \mathbf{0} \ (i \ne j) & & \quad \boldsymbol{\lambda}^T\mathbf{a}_j = 1 \end{array}$$

Since $\min(x_j) = \beta$, $\lambda^T\mathbf{b} = \beta \ge 0$ and. Thus, with $\mathbf{w} = -\lambda$, $\mathbf{w}^T\mathbf{A}=\mathbf{d}^T$ and $\mathbf{w}^T\mathbf{b}\le 0$.

▶ **Karmarka's Algorithm (1984) for large-scale problem (Interior point method)**

- Search in the *strict interior* of the convex feasible region.
- Karmarkar's algorithm is usually more efficient if the problem size is very large

$$\begin{aligned} &\min f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\ &\text{s. t. } \mathbf{Ax}=\mathbf{b} \\ &\qquad \mathbf{x} \ge \mathbf{0} \end{aligned}$$

<Procedure>

1. Start at the centroid ($\mathbf{x}^k$) of the simplex comprising $\mathbf{A}^k\mathbf{x}^k=\mathbf{b}$ (feasible solutions) and project $-\nabla f(\mathbf{x}) = -\mathbf{c}$ onto the intersection of the equality constraints.
$$\tilde{\mathbf{x}}^{k+1} = \mathbf{x}^k + \alpha\mathbf{P}^k(-\mathbf{c}^k) = \mathbf{x}^k - \alpha(\mathbf{I} - \mathbf{A}^{k^T}(\mathbf{A}^k\mathbf{A}^{k^T})^{-1}\mathbf{A}^k)\mathbf{c}^k$$
2. Find α so that only one element of $\tilde{\mathbf{x}}^{k+1}$ becomes zero. And use slightly less value of α, i.e. 0.98α, so that the point lies strictly inside the feasible region.
3. Rescale the variables and transfer $\tilde{\mathbf{x}}^{k+1}$ back to the rescaled centroid. (*Primal affine scaling*)
$$\mathbf{x}^{k+1} = \mathbf{D}_k^{-1}\mathbf{x}^k, \quad \mathbf{c}^{k+1} = \mathbf{D}_k\mathbf{c}^k, \text{ and } \quad \mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{D}_k$$
where $\mathbf{D}_k$ is a diagonal matrix with the elements of $n\tilde{\mathbf{x}}^{k+1}$ as diagonals.
4. If $\mathbf{P}(-\mathbf{c}) \le \varepsilon$ and $\boldsymbol{\lambda} = (\mathbf{AA}^T)^{-1}\mathbf{A})\mathbf{c} \ge 0$ (Lagrange multiplier), then stop. Else, go to Step 1. (For active constraints, λ_i=0 and for inactive, λ_i>0)

cf) Karmarkar actually used a nonlinear transformation

$$\mathbf{x}^{k+1} = \frac{n\mathbf{D}_k^{-1}\mathbf{x}^k}{\mathbf{e}^T\mathbf{D}_k^{-1}\mathbf{x}^k}$$ where $\mathbf{e}$ is an n-dimensional vector of 1's.

Example:

$$\min f(\mathbf{x}) = [1\ 2\ 3]\mathbf{x}$$

$$\text{s. t.}\quad [1\ 1\ 1]\mathbf{x} = 1$$

$$\mathbf{x} \ge \mathbf{0}$$

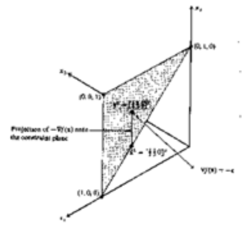
1. Centroid= $\mathbf{x}^0$=[1/3 1/3 1/3]^T , $-\nabla f(\mathbf{x}) = [-1\ \ -2\ \ -3]^T$

$$\mathbf{P} = \mathbf{I} - \begin{bmatrix}1\\1\\1\end{bmatrix}(3)^{-1}\begin{bmatrix}1 & 1 & 1\end{bmatrix} = \begin{bmatrix}2/3 & -1/3 & -1/3\\ -1/3 & 2/3 & -1/3\\ -1/3 & -1/3 & 2/3\end{bmatrix}$$

$$\mathbf{x}^1 = \mathbf{x}^0 + \alpha\mathbf{P}(-\nabla f) = \begin{bmatrix}1/3+\alpha & 1/3 & 1/3-\alpha\end{bmatrix}^T\ (\alpha = 1/3 \rightarrow \tilde{\alpha} = 0.98\alpha\,)$$

For feasibility and **x** to be strict interior, $\tilde{\mathbf{x}}^1 = \begin{bmatrix}1.98/3 & 1/3 & 0.02/3\end{bmatrix}^T$

2. Choose scaling matrix $\mathbf{D}_k$=diag([1.98 1 0.02]), then
   - $\mathbf{x}^k \rightarrow \mathbf{x}^{k+1} = \mathbf{D}_k^{-1}\mathbf{x}^k$ =[0.168 0.333 16.67]^T,
   - $\mathbf{A}^k \rightarrow \mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{D}_k$=[1.98 1 0.02],
   - $\mathbf{c}^k \rightarrow \mathbf{c}^{k+1} = \mathbf{D}_k\mathbf{c}^k$=[1.98 2 0.06]^T
3. Go to Step 1 after checking termination criteria.



<Primal-dual method>

From the constrains of primal and dual problems and the complementary slackness,

$$Ax = b,\ \ x \ge 0,\ \ A^T\lambda + s = c,\ \ s \ge 0,\ \ x_j s_j = 0\ (j = 1,\cdots,n)$$

Main idea of the method is to move through a sequence of strictly feasible primal and dual solutions that come increasingly closer to satisfying the complementary slackness conditions.

$$Ax = b,\ \ x \ge 0,\ \ A^T\lambda + s = c,\ \ s \ge 0,\ \ x_j s_j = \mu\ (j = 1,\cdots,n)$$

The *duality gap*: $c^T x - b^T\lambda = x^T s = n\mu$

Thus, starting from some value of μ , decrease it satisfying the constraints as finding the solution for changes in *x*, *y*, and *s*.

$$A\Delta x = 0,\ \ \Delta x \ge 0,\ \ A^T\Delta\lambda + \Delta s = 0,\ \ \Delta s \ge 0,\ \ s_j\Delta x_j + x_j\Delta s_j = \mu - x_j s_j\ \ (j = 1,\cdots,n)$$

The complementary slackness becomes $S\Delta x + X\Delta s = \mu e - XSe$ where S and X are the diagonal matrices whos diagonals are elements of s and x, respectively.

Then,

$$\Delta\lambda = -(AS^{-1}XA^T)^{-1}AS^{-1}(\mu e - XSe)$$

$$\Delta s = -A^T\Delta\lambda$$

$$\Delta x = S^{-1}(\mu e - XSe) - (S^{-1}X)\Delta s$$

and

$$x_j + \alpha_{\max}\Delta x_j \ge 0,\quad s_j + \alpha_{\max}\Delta s_j \ge 0$$

for all *j*. Then $\alpha_{\max} = \min(\alpha_{primal}, \alpha_{dual})$

where

$$\alpha_{primal} = \min_{\Delta x_j < 0}\left(\frac{-x_j}{\Delta x_j}\right) \text{ and } \alpha_{dual} = \min_{\Delta s_j < 0}\left(\frac{-s_j}{\Delta s_j}\right) \text{ (ratio test)}$$

➔ Simplex method

QUADRATIC PROGRAMMING

Reklaitis G. V., A. Ravindran, and K. M. Ragsdell, "Engineering Optimization : Method and Application," John Wiley and Sons, NY, 1983. (Section 11.2)

I. BASIC PRINCIPLES

▶ **Problem Statement**

min $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + 0.5\ \mathbf{x}^T\mathbf{Q}\mathbf{x}$

subject to $\mathbf{g}(\mathbf{x}) = \mathbf{x} \geq \mathbf{0}$

$\mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}$

▶ **Lagrangian function and Lagrangian multiplier**

Above problem can be rewritten as

min $L(\mathbf{x}, \mu, \lambda) = f(\mathbf{x}) - \mu^T\mathbf{g}(\mathbf{x}) - \lambda^T\mathbf{h}(\mathbf{x})$

subject to $\mu \geq \mathbf{0}$ (λ is unspecified)

where $L(\mathbf{x}, \mu, \lambda)$ is called the *Lagrangian function*, μ and λ are called *Lagrange multipliers*.

▶ **Kuhn-Tucker Conditions (KTC)**

Assume f, $\mathbf{g}$ and $\mathbf{h}$ are differentiable.

The vectors $\mathbf{x}(N\text{x}1)$, $\mu(J\text{x}1)$, $\lambda(K\text{x}1)$ become a candidate for the optimal solution if

i) $\dfrac{\partial L}{\partial \mathbf{x}} = \nabla_x f - \mu^T\nabla_x\mathbf{g}(\mathbf{x}) - \lambda^T\nabla_x\mathbf{h}(\mathbf{x}) = \mathbf{0}$

ii) $\dfrac{\partial L}{\partial \mu} = g(\mathbf{x}) - \mathbf{s} = \mathbf{0} \Rightarrow \mathbf{g}(\mathbf{x}) \geq \mathbf{0}$ (use surplus variable $\mathbf{s}$)

iii) $\dfrac{\partial L}{\partial \lambda} = \mathbf{h}(\mathbf{x}) = \mathbf{0} \Rightarrow \mathbf{h}(\mathbf{x}) = \mathbf{0}$

iv) $\mu \geq \mathbf{0}$ (λ is unspecified)

v) $\mu_j\mathbf{g}_j(\mathbf{x}) = 0$ for $j=1, 2, \cdots, J$ (*complementary slackness condition*).

For complementary slackness condition, if the j-th inequality constraint is binding (active) $\mathbf{g}_j(\mathbf{x}) = 0$ and if j-th inequality constraint is nonbinding (inactive), $\mu_j = 0$.

⇒ Solving i) to v) : *Kuhn-Tucker problem*

- *K-T necessary condition for optimality*

Let f, $\mathbf{g}$ and $\mathbf{h}$ be differentiable, $\mathbf{x}^*$ be a feasible solution.

Assume that $\mathbf{g}(\mathbf{x}^*)$ for active constraints and $\mathbf{h}(\mathbf{x}^*)$ are linearly independent.

cf) Define the effective constraints by $\mathbf{h}^E(\mathbf{x}) = \{h_i(x): h_i(x^*) = 0, \mathrm{i} = 1, \ldots, m\}$, where $\mathbf{h}^E(\mathbf{x})$ is a $(m^E\times 1)$ vector, m^E being the number of effective constraints, with $m^E < n$. Then, the constraint qualification becomes:

constraint qualification (CQ): $rank[\partial\mathbf{h}^E(\mathbf{x}^*)/\partial\mathbf{x}] = m^E$.

→ It is the *regularity conditions* on the feasible region and difficult to verify. However, it is generally acceptable in practice. The CQ states that, at $\mathbf{x}^*$, the m^E effective constraints are "linearly independent around $\mathbf{x}^*$". This guarantees that the implicit function theorem can be used to solve the effective constraints for m^E of the **x** variables and the result substituted back into the objective function.

Then, $\mathbf{x}^*$ and some (μ^*, λ^*) satisfy the Kuhn-Tucker condition.

⇒ When the constraint qualification is not met at the optimum, there may not exist a solution to Kuhn-Tucker problem.

- *K-T sufficient condition for optimality*

Let f be convex, $\mathbf{g}$ be all concave, and $\mathbf{h}$ be linear.

Then $(\mathbf{x}^*, \mu^*, \lambda^*)$ which satisfy the Kuhn-Tucker condition is an optimal solution.

The Kuhn-Tucker condition provides some new information.

- They endogenously treat the constraints which are binding (or non-binding). From the complementary slackness condition, if the j-th constraint is non-binding, $g_j(\mathbf{x}^*)>0$, then the corresponding Lagrange multiplier must be zero, $\lambda_j^* = 0$, $j = 1, \ldots, m$. And if the j-th Lagrange multiplier is positive, $\lambda_j^* > 0$, then the corresponding constraint must be binding, $g_j(\mathbf{x}^*) = 0$, $j = 1, \ldots, n$.
- The Lagrange multipliers are now restricted to be non-negative: $\lambda^* \geq 0$. This reflects the fact that we now have inequality constraints. Indeed, we have seen that the Lagrange multiplier can be interpreted as the marginal value of the constraints. Relaxing an inequality constraint means increasing the feasible set, generating a non-decreasing value of the indirect objective function, and thus a non-negative marginal value of the constraints. In this context, a positive and large Lagrange multiplier means that the corresponding constraint is "very binding" and identifies significant resource scarcity. Alternatively, a small Lagrange multiplier identifies little resource scarcity, as the corresponding constraint is "barely binding." A Lagrange multiplier reaches its lower bound ($\lambda_j^* = 0$) when the j-th constraint is non-binding and becomes irrelevant to the decision.

⇒ When **Q** is positive definite or positive semidefinite for quadratic programming problem, it is sufficient to solve the Kuhn-Tucker conditions to find an optimal solution to the quadratic problem.

▶ **Lagrangian function with equality constraints only**

$$\min f(\mathbf{x}) + \lambda^T(-\mathbf{A}\mathbf{x} + \mathbf{b})$$

Optimality condition if Q is positive definite

$f'(\mathbf{x}) - \mathbf{A}^T\lambda = \mathbf{0}$ and $-\mathbf{A}\mathbf{x} + \mathbf{b} = \mathbf{0}$

$f'(\mathbf{x}) = \mathbf{c} + \mathbf{Q}\mathbf{x}$

$$\Rightarrow \begin{bmatrix} -\mathbf{Q} & -\mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{x}^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -\mathbf{Q} & -\mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix}$$

By *Frobenius-Schur inversion formula*

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & \mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x}^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -\mathbf{Q}^{-1} + \mathbf{Q}^{-1}\mathbf{A}^T(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{Q}^{-1} & \mathbf{Q}^{-1}\mathbf{A}^T(\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}^T)^{-1} \\ (\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{Q}^{-1} & (\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}^T)^{-1} \end{bmatrix}\begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix}$$

Let $\mathbf{A}^* = (\mathbf{A}\mathbf{Q}^{-1}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{Q}^{-1}$ and $\mathbf{H}^* = \mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}^T\mathbf{A}^*$.

$$\begin{bmatrix} \mathbf{x}^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -\mathbf{H}^* & \mathbf{A}^{*T} \\ \mathbf{A}^* & (\mathbf{AQ}^{-1}\mathbf{A}^T)^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix}$$

cf) When **A** and **D** are invertible,

$$(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1}+\mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C})^{-1} & -(\mathbf{A}-\mathbf{BD}^{-1}\mathbf{C})^{-1}\mathbf{BD}^{-1} \\ -(\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B})^{-1}\mathbf{CA}^{-1} & (\mathbf{D}-\mathbf{CA}^{-1}\mathbf{B})^{-1} \end{bmatrix}$$

▶ **Recursive formula**

For a basic feasible solution $\mathbf{x}^0$, $-\mathbf{Ax}^0 + \mathbf{b} = \mathbf{0}$ and $f'(\mathbf{x}^0) = \mathbf{c} + \mathbf{Qx}^0$

$$\mathbf{x}^* = -\mathbf{H}^*\mathbf{c} + \mathbf{A}^{*T}\mathbf{b} = (\mathbf{H}^*\mathbf{Q} + \mathbf{A}^{*T}\mathbf{A})\mathbf{x}^0 - \mathbf{H}^* f'(\mathbf{x}^0)$$

$$\begin{aligned} \boldsymbol{\lambda}^* &= \mathbf{A}^*\mathbf{c} + (\mathbf{AQA}^T)^{-1}\mathbf{b} = (-\mathbf{A}^*\mathbf{Q} + (\mathbf{AQA}^T)^{-1}\mathbf{A})\mathbf{x}^0 + \mathbf{A}^* f'(\mathbf{x}^0) \\ &= (-(\mathbf{AQA}^T)^{-1}\mathbf{AQ}^{-1}\mathbf{Q} + (\mathbf{AQA}^T)^{-1}\mathbf{A})\mathbf{x}^0 + \mathbf{A}^* f'(\mathbf{x}^0) = \mathbf{A}^* f'(\mathbf{x}^0) \end{aligned}$$

II. FLETCHER'S QP ALGORITHM

▶ **QP Algorithm (Fletcher, 1971)**

min $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + 0.5\,\mathbf{x}^T\mathbf{Qx}$ (n decision variables)

subject to $\mathbf{A}^T\mathbf{x} \geq \mathbf{b}$ (m inequality constraints)

$\mathbf{x} \geq \mathbf{0}$

<Procedure>

1. A feasible solution $\mathbf{x}^0$ is given, and q-constraints are tight.
   Let $\mathbf{A}_q$ be the constraints coefficient matrix with q-active constraints.
   Calculate $\mathbf{A}_q^* = (\mathbf{A}_q\mathbf{Q}^{-1}\mathbf{A}_q^T)^{-1}\mathbf{A}_q\mathbf{Q}^{-1}$ and $\mathbf{H}_q^* = \mathbf{Q}^{-1} - \mathbf{Q}^{-1}\mathbf{A}_q^T\mathbf{A}_q^*$ and set k=0.
2. Compute $\mathbf{s}^{(k+1)} = -\mathbf{H}_q^*\nabla f(\mathbf{x}^{(k)})$. If $\mathbf{s}^{(k+1)} = \mathbf{0}$, go to step 4.
3. If $\mathbf{s}^{(k+1)} \neq \mathbf{0}$, compute $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + a^{*(k)}\,\mathbf{s}^{(k+1)}$ and $\nabla f(\mathbf{x}^{(k+1)})$
   where $a^{*(k)} = \min\{1,\ a_p^{(k)}\}$,
   $$\alpha_p^{(k)} = \min\left\{ -\frac{\mathbf{a}_i^T\mathbf{x}^{(k)} - b_i}{\mathbf{a}_i^T\mathbf{s}^{(k+1)}};\ \mathbf{a}_i^T\mathbf{s}^{(k+1)} < 0\ \ (i = q+1,\cdots,m) \right\}$$
   If $a^{*(k)} < 1$, then add p-th column of **A** ($\mathbf{a}_p^{(k)}$) to $\mathbf{H}^*_q$ and $\mathbf{A}^*_q$,
   $$\mathbf{H}_{q+1}^* = \mathbf{H}_q^* - \frac{\mathbf{H}_q^*\mathbf{a}_p\mathbf{a}_p^T\mathbf{H}_q^{*T}}{\mathbf{a}_p^T\mathbf{H}_q^*\mathbf{a}_p}$$
   $$\mathbf{A}_{q+1}^* = \begin{bmatrix} \mathbf{A}_q^* \\ 0 \end{bmatrix} + \frac{\mathbf{a}_p^T\mathbf{H}_q^{*T}}{\mathbf{a}_p^T\mathbf{H}_q^*\mathbf{a}_p}\begin{bmatrix} \mathbf{A}_q^*\mathbf{a}_p \\ 1 \end{bmatrix}$$
   and set k=k+1, q=q+1 and go to step2.
   If $a^{*(k)} = 1$, then set k=k+1 and go to step2.
4. Calculate $\boldsymbol{\lambda}^{(k)} = \mathbf{A}_q^*\nabla f(\mathbf{x}^{(k)})$ and $\lambda_r^{(k)} = \min\left\{\lambda_i^{(k)};\ i = 1,2,\cdots,q\right\}$.

If $\lambda_r^{(k)} \geq 0$ (i.e. all elements of $\boldsymbol{\lambda}^{(k)}$ are nonnegative), Stop (optimum).

If $\lambda_r^{(k)} < 0$, delete r-th row of $\mathbf{A}^*_q$ ($\mathbf{a}_r$) from $\mathbf{H}^*_q$ and $\mathbf{A}^*_q$, and set q=q–1 and go to step2.

$$\mathbf{H}^*_{q-1} = \mathbf{H}^*_q + \frac{\mathbf{a}_r\mathbf{a}_r^T}{\mathbf{a}_r^T\mathbf{Q}\mathbf{a}_r}$$

$$\begin{bmatrix} \mathbf{A}^*_{q-1} \\ 0 \end{bmatrix} = \mathbf{A}^*_q - \frac{\mathbf{A}^*_q\mathbf{Q}\mathbf{a}_r\mathbf{a}_r^T}{\mathbf{a}_r^T\mathbf{Q}\mathbf{a}_r}$$

III. COMPLEMENTARY PIVOT PROBLEMS (LEMKE, 1965)

▶ **Problem Statement**

min $f(\mathbf{x}) = \mathbf{cx} + \mathbf{x}^T\mathbf{Qx}$ (n decision variables)

subject to $\mathbf{Ax} \geq \mathbf{b}$ (m inequality constraints)

$\mathbf{x} \geq \mathbf{0}$

Assume **Q** is symmetric and is positive definite or positive semidefinite.

KTC optimality condition to the above convex quadratic program:

$\mathbf{c} + \mathbf{x}^T(\mathbf{Q} + \mathbf{Q}^T) - \boldsymbol{\mu}^T - \boldsymbol{\lambda}^T\mathbf{A} = 0 \quad \Rightarrow \quad \boldsymbol{\mu} = 2\mathbf{Qx} - \mathbf{A}^T\boldsymbol{\lambda} + \mathbf{c}^T$

$\mathbf{s} = \mathbf{Ax} - \mathbf{b}$

$\mathbf{x},\ \boldsymbol{\mu},\ \boldsymbol{\lambda},\ \mathbf{s} \geq \mathbf{0}$

$\boldsymbol{\mu}^T\mathbf{x} + \mathbf{s}^T\boldsymbol{\lambda} = \mathbf{0}$

Let $\mathbf{w} = \begin{bmatrix} \boldsymbol{\mu} \\ \mathbf{s} \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix}$, $\mathbf{M} = \begin{bmatrix} 2\mathbf{Q} & -\mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} \mathbf{c} \\ -\mathbf{b} \end{bmatrix}$.

Then, $\mathbf{w} = \mathbf{Mz} + \mathbf{q}$ and $\mathbf{w}^T\mathbf{z} = \mathbf{0}$ with $\mathbf{w}, \mathbf{z} \geq \mathbf{0}$.

⇒ *Complementary problem*

- **M** is positive semidefinite since **Q** is positive definite or positive semidefinite.
- If **Q** is set to zero, it becomes an LP.

Definitions:

1. A nonnegative solution (**w**, **z**) to the system of the equation **w** = **Mz** + **q** is called a *feasible solution* to the complementary problem
2. A feasible solution (**w**, **z**) to the complementary problem that also satisfies the complementarity condition is called a *complementary solution*.

- $\mathbf{w}^T\mathbf{z} = \mathbf{0} \Leftrightarrow w_i z_i = 0$ for all i
- The variables w_i and z_i for each i is called a complementary pair of variables.
- If the element of the vector **q** are nonnegative, then there exists an obvious complementary solution given by **w** = **q** and **z** = **0**. (Trivial solution)
- If some elements of the vector **q** are negative, then complementary solution given by **w** = **q** and **z** = **0** would be infeasible. → Introduce a sufficiently large artificial variable z_0 so that $(q_i + z_0)$ become nonnegative.

⇒ A basic feasible solution is given by

$w_i = q_i + z_0$, $z_i = 0$ for all $i = 1, 2, \cdots, n$

$z_0 = -\min(q_i)$

(But $\mathbf{w} = \mathbf{Mz} + \mathbf{q}$ is not met.) → called *almost complementary solution*

<Procedure>

1. To determine the initial almost complementary solution, the variable z_0 is brought into the basis, replacing the basis variable with the most negative value. (Let $q_s = \min q_i < 0$) That is, z_0 replaces w_s from the basis by pivoting.

   $\mathbf{w} - \mathbf{Mz} - z_0\mathbf{e} = \mathbf{q}$

   $\mathbf{w}, \mathbf{z}, z_0 \geq \mathbf{0}$ and $\mathbf{w}^T\mathbf{z} = \mathbf{0}$

   $\mathbf{e}_{(\text{nx1})} = [1, 1, \cdots, 1]^T$

   $\mathbf{T} = [\mathbf{I} \quad -\mathbf{M} \quad -\mathbf{e} \quad \mathbf{q}]$ (initial tableau)

2. In order to maintain the complementarity, either one of w_s and z_s should remain as a basic variable. So in the next tableau, the complement of the basic variable that just left the basis in the last tableau should become a basic variable. (*complementary rule*) In order to maintain the nonnegativity of the basic feasible solution, use *minimum ratio test* to determine the variable to leave the basis.

$$\frac{q_k}{m_{ks}} = \min\left\{\frac{q_i}{m_{is}};\ i = 1, 2, \cdots, n \text{ and } m_{is} > 0\right\}$$

   That is, z_s replaces w_k from the basis by pivoting.

4. Since w_k left the basis, the variable z_k is brought into the basis by the complementary rule and the basis changes as before. If the minimum ratio is obtained in row *s*, and z_0 leaves the basis, the resulting basic solution after performing the pivot operation is the *complementary solution*. If the minimum ratio test fails, since all the coefficients in the pivot column are nonpositive, this implies no solution to the complementary problem exists. In this case, we say that the complementary problem has a *ray solution*. (the given linear or quadratic program has no solution)

Example:

min $f(\mathbf{x}) = -6x_1 + 2x_1^2 - 2x_1 x_2 + 2x_2^2$

subject to $-x_1 - x_2 \geq -2$ and $x_1, x_2 \geq 0$

For $f(\mathbf{x}) = \mathbf{cx} + \mathbf{x}^T\mathbf{Qx}$, $\mathbf{c} = [-6 \quad 0]^T$, $\mathbf{Q} = [2 \quad -1; -1 \quad 2]$.

For $\mathbf{Ax} \geq \mathbf{b}$, $\mathbf{A} = [-1 \quad -1]$, $\mathbf{b} = -2$, $\mathbf{w} = [\mu_1 \quad \mu_2 \quad s]^T$, $\mathbf{z} = [x_1 \quad x_2 \quad \lambda]^T$.

$$\mathbf{M} = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & 1 \\ -1 & -1 & 0 \end{bmatrix} \text{ and } \mathbf{q} = \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix}$$

Basis	w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
w_1	**1**	**0**	**0**	**-4**	**2**	**-1**	**(-1)**	**-6**
w_2	**0**	**1**	**0**	**2**	**-4**	**-1**	**-1**	**0**
w_3	**0**	**0**	**1**	**1**	**1**	**0**	**-1**	**2**

An almost complementary solution is obtained by replacing w_1 by z_0.

Basis	w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
z_0	**-1**	**0**	**0**	**4**	**-2**	**1**	**1**	**6**
w_2	**-1**	**1**	**0**	**(6)**	**-6**	**0**	**0**	**6**
w_3	**-1**	**0**	**1**	**5**	**-1**	**1**	**0**	**8**

From the complementarity, z_1 has to be a basic since w_1 became nonbasic.

The ratio test: q_i/m_{si} = 6/4, 6/6, 8/5. Therefore choose w_2.

Basis	w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
z_0	-1/3	-2/3	0	0	2	1	1	2
z_1	-1/6	1/6	0	1	-1	0	0	1
w_3	-1/6	-5/6	1	0	(4)	1	0	3

From the complementarity, z_2 has to be a basic since w_2 became nonbasic.
The ratio test: q_i/m_{si} = 2/2, -1/1, 3/4. Therefore choose w_3.

Basis	w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
z_0	-1/4	-1/4	-1/2	0	0	(1/2)	1	1/2
z_1	-5/24	-1/24	1/4	1	0	1/4	0	7/4
z_2	-1/24	-5/24	1/4	0	1	1/4	0	3/4

From the complementarity, z_3 has to be a basic since w_3 became nonbasic.
The ratio test: q_i/m_{si} = 1, 7, 3. Therefore choose z_0.

Basis	w_1	w_2	w_3	z_1	z_2	z_3	z_0	q
z_3	-1/2	-1/2	-1	0	0	1	2	1
z_1	-1/12	-1/12	1/2	1	0	0	-1/2	3/2
z_2	-1/12	-1/12	1/2	0	1	0	-1/2	1/2

Since z_0 left the basis, the *complementary solution is obtained*.
$z_1= x_1=3/2$, $z_2= x_2=1/2$, $z_3=\lambda=1$, $w_1= w_2= w_3=0$ and $f(\mathbf{x}^*)= -11/2$.